Dynamic Legislative Policy Making*

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Abstract

We prove existence of stationary Markov perfect equilibria in an infinite-horizon model of legislative policy making in which the policy outcome in one period determines the status quo in the next. We allow for a multidimensional policy space and arbitrary smooth stage utilities. We prove that all such equilibria are essentially in pure strategies and that proposal strategies are differentiable almost everywhere. We establish upper hemicontinuity of the equilibrium correspondence, and we derive conditions under which each equilibrium of our model determines a unique invariant distribution characterizing long run policy outcomes. We illustrate the equilibria of the model in a numerical example of policy making in a single dimension, and we discuss extensions of our approach to accommodate much of the institutional structure observed in real-world politics.

1 Introduction

Political interaction in modern democracies counts among the most complex phenomena subjected to scientific inquiry, and practical considerations dictate that we attempt to accommodate this complexity in formal political modeling. Doing so would appear essential, for example, for the detailed analysis of the effects of public policy and the design of constitutions. In this spirit, we study policy making within a legislative body or, more generally, a democratic government in which policy initiatives are systematically subjected to review by political actors with well-defined authority. Our goal is to develop a model of policy making that (i) accounts for the multidimensional nature of public policy and idiosyncratic details of policy preferences, (ii) captures the ongoing nature of policy making, and (iii) allows for the kinds of random shocks (e.g., on preferences and the environment) to which political interaction is subjected over time. We provide a benchmark model that satisfies these desiderata and allows us to address the central theoretical issues confronting applications. The model is intentionally austere, in that we do not incorporate the rich spectrum of political institutions observed in the real world, but our approach is very general: we conclude with a discussion of how our results extend to an institutionally detailed version of the model. And although we are motivated by the application to legislatures and democratic politics, the issues we address are fundamental and would arise in a host of dynamic bargaining contexts, such as wage

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negotiation in labor markets, or collusion among members of a cartel, or deliberations among a board of directors, or treaty talks among states.

We lay the theoretical groundwork for future applications by establishing the existence of equilibria satisfying a number of desirable regularity properties in a class of models satisfying the objectives (i)–(iii) identified above. We consider a fully dynamic model of legislative policy making in which each period begins with a status quo policy and the random draw of a legislator, who proposes any feasible policy, which is then subject to an up or down vote. The policy outcome in that period is the proposed policy if it receives the support of a "decisive" coalition of legislators, the status quo otherwise, and the status quo in the next period is determined by the outcome that prevails in the current period. This process continues ad infinitum. Thus, the path of play in our model generates an infinite sequence of policies over time, and in equilibrium, legislators must anticipate future policy outcomes. When voting on a proposal, a legislator must compare the stream of policies that would be engendered by the proposed policy with the stream that would be generated by the status quo. And legislators must select their proposed policies optimally in light of the future policies they entail, while factoring in the prospects for a proposed policy to garner the support of a decisive coalition. We deduce the existence of stationary equilibria in pure strategies, and moreover, we show that *all* stationary equilibria are essentially pure. In fact, equilibria are strict in the sense that proposers almost always have unique optimal proposals, a critical property for the computation of equilibria. Continuation values of the legislators are differentiable, and equilibrium proposal strategies are differentiable almost everywhere, making possible the use of calculus techniques in characterizing optimal proposals. We prove a general result on upper hemicontinuity of the equilibrium correspondence with respect to the parameters of the model, including the policy space itself. Finally, we give conditions under which each equilibrium admits a unique invariant distribution with desirable ergodic properties, providing an unambiguous prediction of long run policy outcomes in the model.

We do not impose specific assumptions about the policy space or functional forms for legislators' utility functions. Instead, we allow the set of alternatives to be a very general subset of any finite-dimensional Euclidean space defined by smooth feasibility constraints, and we assume smooth stage utility functions but do not impose any further restrictions on preferences. Thus, we capture standard models with resource and consumption constraints, such as the classical spatial model of politics, economic environments, and distributive models in which a fixed surplus is allocated to the legislators, and we obtain even a finite policy space as a special case. We incorporate uncertainty about future policy preferences and future effects of policy with the assumption that at the end of each period: next period's status quo is realized as the sum of the current period's policy outcome and an arbitrarily small stochastic shock, and legislators' preferences next period are subject to arbitrarily small publicly observed stochastic shocks. The first of these two assumptions captures the fact that policy instruments are often pegged to the realization of random variables. For example, legislators may care about the real minimum wage, which is determined through a nominal minimum wage and the realization of the current price level, the latter being a random variable for our purposes. In fact, we would argue that because no legal document can account for all possible contingencies, all policy is implicitly subject to unforeseeable shocks: the implementation of any policy codified in law will ultimately be subject to review by courts, interpretation by administrators, and the vagaries of the policy environment. Thus, a particular policy decision in the current period is likely to effectively result in a different, albeit correlated, policy implemented in future periods. The second assumption captures the possibility of mild idiosyncratic deviations from the underlying systematic preferences of legislators from period to period. These shocks can be viewed as a reduced form representation of uncertainty about the preferences of constituent voters or about other aspects of the legislators' electoral environments.

The equilibrium behavior of legislators in our model is potentially highly complex, owing to the vast multiplicity of histories in the game. At a minimum, proposal strategies must depend on the current status quo and preference parameters, and voting strategies must depend additionally on the policy proposed. It is therefore natural to focus on stationary Markov perfect equilibria, a refinement that precludes more complicated forms of history-dependence. Due to their relative simplicity, such strategies minimize the difficulty of strategic calculations and may therefore possess a focal quality. From a practical econometric point of view, moreover, stationarity is critical for the identification of empirical models (cf. Ericson and Pakes (1995) and Aguirregabiria and Mira (2007)).[1] Existence of stationary equilibria in general dynamic games is difficult, however, for discontinuities can arise due to expectations of future play of the game: if players use discontinuous strategies, then a small change in one player's action in the current period may lead to a large response by other players, creating a jump in the discounted sum of payoffs even if stage payoffs are continuous. It is customary in the literature on stochastic games to gain traction on existence by adding noise to the transition from the current state to next period's state and imposing continuity assumptions on transition probabilities. In our model, uncertainty about next period's status quo and future preferences of legislators plays a similar, though diminished, role. These two types of noise, the common shock to current policy and the idiosyncratic shock to each legislator's preferences, confer rather distinct analytical benefits: the idiosyncratic component is critical for uniqueness of best responses, while the common component is used to obtain needed continuity and compactness conditions. A similar decomposition of noise can be found in the dynamic industrial organization literature, as in Aguirregabiria and Mira (2007) and Doraszelski and Satterthwaite (2007).[2] As we discuss in the next section, however, our formulation of noise is not sufficient for the model to fulfill the standard continuity assumptions on transition probabilities, and we cannot obtain existence "off the shelf."

Given the complexity of our model, analytical solutions are likely to be out of reach in many interesting applications, and we therefore add a fourth item to the desiderata (i)–(iii) stated above: we seek a model that (iv) permits analysis of equilibrium by numerical methods. We prove existence of equilibrium, a necessary condition for the meaningful application of numerical methods, but existence alone is not sufficient for numerical analysis, as dynamic games with a continuum of actions can present formidable challenges to computation of equilibrium if players use mixed strategies. In this respect, our characterization results take on added significance. We know that all equilibria of our model are essentially pure, and in fact we show that proposers almost always have unique best responses, removing an important obstacle to the application of numerical techniques. The significance of pure strategy equilibria for the computational tractability of dynamic games is highlighted by Herings and Peeters (2004) and Nowak (2007) and is emphasized by Doraszelski and Satterthwaite (2007) in the context of a dynamic oligopoly model. We take an additional step toward a practical algorithm for numerical analysis of our model by means of an increasing sequence of finite grids on the policy space. Because we prove upper hemicontinuity while allowing the policy space to vary, an implication of our results is that if we compute stationary equilibria for each finite grid, then the ensuing sequence of equilibrium continuation values will have an accumulation point, and this will correspond to an equilibrium of the continuum model.

[1] See Maskin and Tirole (2001) for an elaboration of these points and further grounds for interest in stationary equilibria.

[2] The addition of noise to dynamic models has also been fruitful in such applications as the study of intergenerational transfers in growth economies. See, e.g., Bernheim and Ray (1989) and Nowak (2006).

We demonstrate this approach in a simple example in which five legislators must choose policies in an interval of the real line. We compute a stationary equilibrium that illustrates how slight asymmetries in stage utilities can lead to qualitative differences in equilibrium behavior, and we verify that the unique ergodic distribution piles probability mass close to the ideal point of the median voter.

Our paper is related to the work of Baron and Ferejohn (1989) and the political science literature spawned from it, but with an important difference: their model ends with the proposed allocation of surplus if a majority of legislators accept the proposal, which occurs, in equilibrium, in the first period of the game. Thus, the model can be used to examine policy choices across legislative sessions by simply repeating the bargaining game each session, but this is appropriate only if policies remain in place for a single session with an exogenously fixed default outcome at the beginning of the next. This is often the case in budgetary negotiations, but the model is inadequate for the analysis of continuing legislation, where policy remains in place for the indefinite future and endogenously determines the status quo in subsequent negotiations. In seminal work on the endogenous status quo model, Baron (1996) considers a model in which legislators have single-peaked stage utilities over a one-dimensional policy space and must choose a sequence of policies over time, each period's policy outcome becoming the status quo in the next. He proves that stationary equilibrium policy outcomes converge to the ideal point of the median voter over time, consistent with our numerical example of one-dimensional policy-making, though our model generates richer dynamics due to the presence of noise. The model has been extended to special multidimensional settings by Kalandrakis (2004c, 2005a), Fong (2005), Cho (2005), and Battaglini and Coate (2007b,a), who give constructive proofs of equilibrium existence relying on the particular structure of their models.

In Section 2, we give a more thorough review of the bargaining literature in political economy and the literature on stochastic games, as well as the related literature in dynamic industrial organization. In Section 3, we present the model formally and describe our solution concept. In Section 4, we state our existence and characterization results. In Section 5, we present a numerical example. We conclude with a discussion of extensions of the model in Section 6, and we collect all proofs in Appendix A.

2 Literature Review

Before turning to the analysis, we first give a more in-depth review of the literature on bargaining, as it relates to legislative modeling, the literature on existence of stationary Markov perfect equilibrium in stochastic games, and the related literature in dynamic industrial organization.[3]

Bargaining Most of the existing work on bargaining considers an infinite-horizon game where in each period one agent makes a proposal and that proposal is either accepted, in which case the game ends with the proposed outcome, or rejected, in which case bargaining continues for at least one more round. This literature begins with the work of Rubinstein (1982) on two-person, alternating-offer bargaining, which is modified by Binmore (1987) to allow for a randomly determined proposer. The model was extended to cover legislative politics by Baron and Ferejohn (1989), who allow for an arbitrary number of legislators and assume a simple majority is required for a proposal to pass. As with Rubinstein's and Binmore's work, the subject of bargaining is the

[3]Our model also has connections, though not as tight, to the macroeconomic literature on quasi-geometric discounting, where dynamic decision making is modeled as a game between present and future selves (cf. Krusell and Smith (2003)). Here, the extra discounting on future utility acts somewhat like the probability of a "self" being recognized to make a proposal. We thank Tony Smith for this observation.

allocation of a fixed surplus, often interpreted as pork barrel spending in the legislative literature.

A substantial literature cutting across economics and political science has grown from these papers. For example, Baron (1991) examines the case of a two-dimensional set of alternatives, three or four voters with quadratic preferences, and voting by majority rule. Merlo and Wilson (1995) prove uniqueness of stationary equilibrium, assuming unanimity rule and allowing the amount of the surplus to vary stochastically over time. Eraslan (2002) proves uniqueness of stationary equilibrium in the original Baron-Ferejohn model. Banks and Duggan (2000) prove existence and examine connections to the core of the cooperative voting game in a version of the model with general set of alternatives, preferences, and voting rule. Kalandrakis (2004b) gives a simplified proof of existence using a characterization of equilibrium in terms of the solutions to a finite number of equalities and inequalities. Kalandrakis (2006a) examines regularity of the general bargaining model for generic discount factors. While all of the previous work implicitly assumes that delay is bad for the agents, Banks and Duggan (2006) allow for an arbitrary status quo, re-establish results from the earlier framework, and provide a new analysis of the possibility of delay. Cho and Duggan (2003) prove uniqueness of stationary equilibrium in the one-dimensional model with quadratic utilities, and Cho and Duggan (2005) prove an asymptotic median voter theorem in the one-dimensional bargaining model without stationarity. This class of models has found numerous applications to legislative policy-making,[4] but while these applications capture some dynamic aspects of politics, they uniformly assume that the game ends once a proposal is accepted.

A small literature considers the effects of endogenizing the status quo: each period begins with a status quo, then one agent makes a proposal and that proposal is either accepted, in which case it becomes the current policy and the status quo for the next period, or rejected, in which case the current status quo remains in place until the beginning of next period. Whether the current period's proposal is accepted or rejected, the process is repeated next period, and so on. There are currently no general results for this model, though there are constructions of stationary equilibria in special cases. Baron (1996) analyzes the one-dimensional version of the model with single-peaked stage utilities. Kalandrakis (2004a, 2005a) establishes existence and continuity properties of equilibrium strategies in the distributive model, obtains a fully strategic version of McKelvey's (1976; 1979) dictatorial agenda setting in that setting, and studies the composition of equilibrium coalitions and the effect of risk-aversion on equilibrium.[5] Baron and Herron (2003) give a numerical calculation of equilibrium in a three-legislator, finite-horizon model. Fong (2005) considers a three-legislator model in which policies consist of locations in a two-dimensional space and allocations of surplus. Cho (2005) analyzes policy outcomes in a similar environment but with a stage game emulating aspects of parliamentary government. Similar in spirit to the above, Battaglini and Coate (2007b) characterize stationary equilibria in a model of public good provision and taxation with identical legislators and a stock of public goods that evolves over time. Battaglini and Coate (2007a) consider a dynamic model of public spending and taxation in which the state variable is the amount of public debt. All of the above analyses of stationary equilibria consist of explicitly constructing equilibrium strategies, which, given the dependence of proposals on the status quo, can be extremely complex.

A number of related papers diverge in various ways from the above literature and our model. Bernheim et al. (2006) analyze a model of a single policy choice in which the proposal on the floor is subject to change over time, and after a fixed number of rounds, the implemented policy is

[4]See, for example, Diermeier et al. (2003), Jackson and Moselle (2002), Kalandrakis (2004a, 2005b, 2006b), McCarty (2000), and Merlo (1997).

[5]In contrast, Epple and Riordan (1987) allow for history dependent strategies and derive folk theorem results in the distributive model.

determined by a final up or down vote between the proposal offered in the last round and the previous proposal on the floor. The authors assume a finite policy space and strict preferences over policies for all legislators, so that backward induction yields a unique equilibrium outcome. They then extend the model to a finite number of policy choices, with the finite horizon again permitting backward induction.[6] Penn (2005) considers a dynamic voting game with randomly generated policy proposals and probabilistic voting on these proposals. Lagunoff (2005a,b) investigates a class of stochastic games that incorporate a social choice solution concept and analyzes endogenous political institutions. Finally, Gomez and Jehiel (2005) consider a class of stochastic games and characterize efficiency properties of equilibrium when players are patient. Unlike our model, they assume a finite number of states and transferable utility.

Stochastic Games Existence of stationary Markov perfect equilibrium is a central issue in the literature on stochastic games beginning with Shapley (1953), who proved existence for finite, two-player, zero-sum games. Existence in general finite stochastic games follows from the straightforward application of Kakutani's fixed point theorem in finite dimensions (cf. Rogers (1969) and Sobel (1971)). Haller and Lagunoff (2000) prove that the set of stationary Markov perfect equilibria in finite games is generically finite, and Herings and Peeters (2004) develop an algorithm for computation of equilibrium and use homotopy arguments to show that the number of equilibria is odd. General results on existence have been elusive and have relied on the imposition of relatively special structure or departures from the concept of stationary equilibrium.[7] All of the known results rely on fairly strong assumptions on transition probabilities. Letting s denote a state and a denote a profile of actions, a transition probability is a measurable mapping $\mu_t(\cdot|s, a)$ from state-action pairs to a probability measure on the set of states. We next list, in increasing strength, some assumptions used in the literature, where we refer to the total variation norm on probability measures.

(A1) μ_t is set-wise continuous in a,[8]

(A2) μ_t is norm-continuous in a,

(A3) μ_t is norm-continuous in a and absolutely continuous with respect to some fixed probability measure ν_t,

(A4) μ_t is norm-continuous in a and absolutely continuous with respect to a fixed, non-atomic probability measure ν_t,

(A5) μ_t has a density $f(s'|s, a)$ with respect to Lebesgue measure that is continuous with respect to a.

It is well-known that even the weakest of the above assumptions, (A1), is inconsistent with deterministic transitions when action sets are uncountably infinite.

In finite-horizon stochastic games, Rieder (1979) (see also Chakrabarti (1999)) proves existence of Markov perfect equilibrium under (A1). By incorporating time in the state variable of a finite-horizon game, we may in fact view Rieder's equilibrium as stationary. Assuming (A1), Whitt (1980) and Escobar (2006) prove existence of stationary Markov perfect equilibria for the

[6] A further difference is that Bernheim et al. (2006) allow for persistent policy programs, which can determine trajectories of policies in future periods.

[7] Dutta and Sundaram (1998) provide a lucid review of much of the literature on stochastic games and the problem of existence of Markov Equilibrium.

[8] That is, for each measurable set Z of states, $\mu(Z|s, a)$ is continuous in a.

case of a countable state space. Under strong continuity assumptions on the transition probability, akin to (A5), Amir (1996, 2002)), Curtat (1996), and Nowak (2007) prove existence of stationary Markov perfect equilibria in games possessing strategic complementarities.[9] Other results have been obtained by weakening stationarity or considering weaker notions of equilibrium. Chakrabarti (1999) proves existence of (possibly non-stationary) Markov perfect equilibria in games satisfying (A3), and Mertens and Parthasarathy (1987, 1991) drop the assumption of absolute continuity and obtain existence of equilibria that are nearly Markovian.[10] Increasing (A3) to (A4), Chakrabarti (1999) proves existence of a stationary equilibrium, but now in semi-Markov perfect strategies.[11] Dutta and Sundaram (1998) give a simple proof of the existence of (possibly non-stationary) Markov perfect ϵ-equilibria under (A1), whereas Nowak (1985) increases (A1) to (A4) and obtains a Markov perfect ϵ-equilibrium in stationary strategies. Himmelberg et al. (1976) prove existence of stationary p-equilibria assuming finite action sets.[12] Finally, Nowak and Raghavan (1992) prove existence of stationary Markov perfect equilibria with public randomization under (A4), and Duffie et al. (1994) add mutual absolute continuity of transition probabilities and show that the equilibrium induces an ergodic process.

There at least three difficulties in applying known results on stochastic games to our legislative model. The first is simply that there is no general existence result for stationary Markov perfect equilibria in stochastic games with a continuum of actions and states: current work in that setting weakens stationarity or relaxes the exactness of optimality conditions or allows for correlated strategies. But we seek truly stationary strategies such that legislators optimize at all states, without the availability of a public randomization device. The next challenge stems from the deterministic element inherent in the structure of legislative procedure, which when modelled naturally violates (A1) and the other stronger conditions used on transition probabilities. In describing our legislative model as a stochastic game, the state variable must include all relevant details of the game when legislators take actions, whether proposing policy or voting over proposals. Thus, the state must specify when the legislature is in a proposal stage or a voting stage. In a proposal stage, the state must also specify the proposer and status quo, and in a voting stage the state must specify the proposed policy and the status quo. The transition from a voting stage to the subsequent proposal stage is not problematic, for action sets in the voting stage are finite, and the transition is trivially continuous following the votes of legislators. The problem is the fact that the proposer's action (the policy proposal) precisely determines the state in the subsequent voting stage, inevitably violating (A1). Adding noise to the model in the form of uncertainty about the future status quo and policy preferences of legislators does not eliminate this problem, which we take to be an inherent feature of legislative policy-making. Our model violates the standard continuity assumptions, so that even a general existence result using the weakest of these assumptions, if one were proved, would not apply. We are nevertheless able to establish existence of stationary equilibria satisfying a number of desirable technical properties.

The final difficulty arises from the desirability of pure strategy equilibria, discussed above. In the extant literature, Whitt (1980) and Escobar (2006) give sufficient conditions for existence of pure strategy equilibria by imposing restrictive conditions on payoffs that are satisfied, for example,

[9] Other work restricts the way in which players' actions affect each others' payoffs, e.g., Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), and Horst (2005).

[10] Players strategies in period t can depend not only on the current state s_t but the previous state s_{t-1} as well.

[11] That is, players may condition on the current state and the state in the previous period, and the nature of that conditioning is constant over time.

[12] Here, p is a probability measure on states, and a p-equilibrium is a strategy profile such that players optimize at all but perhaps a set of states with p-measure zero.

when players are sufficiently impatient. Amir (1996, 2002), Curtat (1996), and Nowak (2007) derive pure strategy equilibria using lattice-theoretic methods, while the latter author also gives conditions based on concavity of the stage game and a decomposition of the transition probability. Horst (2005) gives a sufficient condition that limits the dependence of a player's payoff on actions of others. None of these results apply in the context of our model.

Industrial organization A strand of literature in industrial organization, beginning with Ericson and Pakes (1995), has studied the dynamics of entry, exit, and investment in a framework with some similarities to our legislative policy making model. Ericson and Pakes (1995) consider an industry, essentially, with a finite number of states characterizing the profitability of all firms. Each period begins with a state, and this determines gross profits for that period. Active firms must choose a level of investment and whether to remain in the industry, while inactive firms must decide whether to enter the industry. Then next period's state is randomly drawn from a distribution that depends on the state and investment decisions in the current period. Doraszelski and Satterthwaite (2007) observe that equilibria in the Ericson-Pakes model may require mixed strategies. The latter authors assume incomplete information about scrap values and setup costs of firms, which are distributed iid, to obtain pure entry/exit strategies, and they impose restrictions on the transition probability to obtain pure investment strategies. Thus, their model involves a decomposition of noise into a common component (the shock to the productivity state) and an idiosyncratic component (scrap values and setup costs), similar to ours. The nature of the idiosyncratic noise is quite distinct, however. In Doraszelski and Satterthwaite (2007), scrap values and setup costs are private information, and their purification of entry/exit decisions is in the spirit of Harsanyi (1973). In contrast, our model is characterized by symmetric information: preference shocks are realized at the beginning of the period and are publicly observed, and therefore in equilibrium, a legislator's optimal proposal will depend on not just her own preference shock but on the shocks of all other legislators. Generic uniqueness of optimal proposals in our model follows from the observation that the constraints in a legislator's optimal proposal problem will depend on the shocks of others but not on the shock of the legislator herself.

In recent applied work, Aguirregabiria and Mira (2007) consider an empirical model in which there is a finite set of states and firms make discrete choices that determine profits and the distribution of states in the next period. In addition, the current period's profits are subject to idiosyncratic, iid shocks. Thus, the model involves a decomposition of noise similar to ours. As with Doraszelski and Satterthwaite (2007), however, the idiosyncratic shocks are private information. Furthermore, the idiosyncratic shocks of Aguirregabiria and Mira (2007) are shocks to the actions of firms, rather than to economic outcomes, and thus generic uniqueness of best responses is immediate. In our model, preference shocks are applied to the utility of the current policy outcome, and not to the actions (proposals and votes) leading to that outcome. Finally, although idiosyncratic shocks are continuously distributed in Doraszelski and Satterthwaite (2007) and Aguirregabiria and Mira (2007), the set of public states is finite. This greatly simplifies the problem of imbedding continuation values within a compact space, an important step in obtaining existence of stationary Markov perfect equilibrium. An additional role of the common component of noise in our model, not needed in the above work, is to achieve that imbedding.

3 Legislative Model

Framework We posit a finite set N of legislators, $i = 1, \ldots, n$, who bargain over a set of feasible policies in each period $t = 1, 2, \ldots$. Legislative interaction proceeds as follows. A status quo policy $q_t \in \Re^d$ and a vector $\theta = (\theta_1, \ldots, \theta_n) \in \Re^{nd}$ of preference parameters are realized and publicly observed. A legislator $i \in N$ is drawn at random, with probabilities $p_1, \ldots, p_n$, to propose a policy $y \in X \cup \{q_t\}$, where $X \subseteq \Re^d$ represents the set of feasible policies. The legislators vote simultaneously to accept y or reject it in favor of the status quo q_t. The proposal passes if a coalition $C \in \mathscr{D}$ of legislators vote to accept, and it fails otherwise, where $\mathscr{D}$ is a collection of coalitions described later. The policy for period t, denoted x_t, is y if the proposal passes and is q_t otherwise. Each legislator j receives utility $\hat{u}_j(x_t, \theta_j)$, where $\theta_j \in \Re^d$ is the legislator's utility shock. Finally, the status quo q_{t+1} for period $t+1$ is drawn from the density $g(\cdot|x_t)$, a new vector $\theta = (\theta_1, \ldots, \theta_n)$ of preference shocks is drawn from the density $f(\cdot)$ and publicly observed, and the above procedure is repeated in period $t+1$. Payoffs in the dynamic game are given by the expected discounted sum of stage utilities, as is standard, and we denote the discount factor of legislator i by $\delta_i \in [0, 1)$.

We represent a general voting rule by a nonempty collection $\mathscr{D} \subseteq 2^N \setminus \{\emptyset\}$ of *decisive coalitions* satisfying only the minimal monotonicity requirement that if one coalition is decisive, and we add legislators to that coalition, then the larger coalition is also decisive. Formally, we assume that if $C \in \mathscr{D}$ and $C \subseteq C' \subseteq N$, then $C' \in \mathscr{D}$. This allows us to capture majority rule in the obvious way, $\mathscr{D} = \{C \subseteq N : |C| > \frac{n}{2}\}$, and we obtain as special cases other common voting rules, such as unanimity rule, $\mathscr{D} = \{N\}$, or any quota rule, $\mathscr{D} = \{C \subseteq N : |C| \geq q\}$, where $q \in [0, N]$, or even dictatorship, $\mathscr{D} = \{C \subseteq N : i \in C\}$, where i is the dictator. But this structure allows us to capture more complex voting rules as well. For example, if a player i has a veto, we incorporate this by demanding that $i \in C$ for all $C \in \mathscr{D}$. See Banks and Duggan (2000, 2006) for examples of how a bicameral system with executive veto, as in the US system, is obtained, along with examples of more esoteric rules of the US Congress.

We impose a number of regularity conditions on the model. We assume that the set of feasible policies, X, is cut out by a finite number k of functions $h_\ell \colon \Re^d \to \Re$ indexed by $K = \{n+1, \ldots, n+k\}$. We decompose these into k^{in} inequality constraints, K^{in}, and k^{eq} equality constraints, K^{eq}, where $k = k^{in} + k^{eq}$. Thus, we have

$$X = \{x \in \Re^d : h_\ell(x) \geq 0, \ell \in K^{in}, h_\ell(x) = 0, \ell \in K^{eq}\}.$$

We assume that X is compact, and that h_ℓ is r-times continuously differentiable for all $\ell \in K$, where $r \geq \max\{2, d\}$ is the order of differentiability assumed in the model.[13] For technical reasons, we impose the weak condition that for all $x \in X$, $\{Dh_\ell(x) : \ell \in \overline{K}\}$ is linearly independent, where $\overline{K}$ is the subset of $\ell \in K$, including equality constraints, such that $h_\ell(x) = 0$. With these assumptions, we capture standard models with resource and consumption constraints, such as the classical spatial model of politics, public good economic environments, and distributive models in which an amount of surplus is to be allocated among the legislators' districts. Moreover, equality constraints allow us to capture quite general manifolds, and in particular we obtain an arbitrary finite set $X \subseteq \Re^d$ of policies as a special case.[14]

The presence of preference shocks in the model allows us to capture uncertainty about the legislators' future policy preferences. We assume that stage utilities are given by $\hat{u}_i(x, \theta_i) = u_i(x) + \theta_i \cdot x$, where $u_i \colon \Re^d \to \Re$ is r-times continuously differentiable. In the special case of negative quadratic utility, i.e., $u_i(x) = -||x - \hat{x}_i||^2$ where $\hat{x}_i \in \Re^d$ is a fixed ideal point, the preference

[13] Of course, we allow $r = \infty$.

[14] We do so using suitably "oscillating" equality constraints. We can, for example, isolate a grid on $[0, 1]^d$ by using trigonometric functions, as in $\{x \in \Re^d : \sin(2\pi x_i \alpha) = 0, i = 1, \ldots, d\}$, for appropriate α. We exploit this possibility in the numerical analysis of Section 6.

shock is equivalent to adding a noise term to the ideal point of legislator i,[15] while in the case of linear utility, the shock is merely a perturbation of a legislator's gradient. We choose the linear form because it respects the standard convexity and continuity assumptions in the literature and is the simplest way of introducing well-behaved shifts of the indifference curves of legislators: it is a trivial matter to extend our results to the model with a more complex parameterization, as long as it contains a linear component.[16] With regard to the distribution of preference shocks, we assume that the vector $\theta = (\theta_1, \ldots, \theta_n)$ is distributed according to a density f with support contained in an open set $\Theta \subseteq \Re^{nd}$, and we further assume a compact set $\tilde{X} \supseteq X$ and a bound c such that $|u_i(x) + \theta_i \cdot x|f(\theta) \leq c$ for all $i \in N$, all $\theta \in \Theta$, and all $x \in \tilde{X}$. Since $\tilde{X}$ is compact, it has finite Lebesgue measure, which we denote by a.

The noise on the status quo captures the idea that legislators are uncertain about the way policy decisions today will be implemented in the future. We assume that the density $g\colon \Re^d \times \Re^d \to \Re$, with values $g(q|x)$, is jointly measurable in (q, x), and that for all x, the support of the density $g(\cdot|x)$ lies in the compact set $\tilde{X}$. We do not assume that the support of $g(\cdot|x)$ lies in X, though of course we allow it. Furthermore, we assume a bound b such that for all q, we have: $g(q|x)$ is r-times continuously differentiable in x; if $r < \infty$, then all derivatives of order $1, \ldots, r$ are bounded in norm by b, and the r-th derivative of $g(q|x)$ with respect to x is Lipschitz continuous with modulus b; and if $r = \infty$, then derivatives of all orders $1, 2 \ldots$ are bounded in norm by b. It is a trivial matter to extend the model to allow the density g of next period's status quo to depend on the current status quo, in addition to the current policy outcome.

Our approach to existence involves the addition of noise to policy outcomes and legislator utilities, but we emphasize that the status quo and the utility shocks at the beginning of a period t are commonly known, so that a proposer knows whether any given policy will pass or fail if proposed. Furthermore, once a vote is taken, the policy outcome is pinned down for period t: the legislators know, conditional on the outcome of voting, what the policy outcome in the current period will be, and a new status quo is drawn for period $t + 1$ only after legislators receive their period t utilities from outcome x_t. Thus, our formulation of noise in the model is consistent with the view that while legislators are completely informed in the current period, there is at least some uncertainty about future policy preferences and the policy environment. We view these as natural modeling assumptions. In any case, the variance of the densities f and $g(\cdot|x)$ may be arbitrarily low, with the sole caveat that the variance of $g(\cdot|x)$ must be bounded above zero uniformly across x. Thus, we allow for the selection of preference shocks and the status quo to be arbitrarily close to deterministic, so that the element of noise in the model can be made negligible from a substantive standpoint.

An alternative would be to model uncertainty in the current period as well, so that a legislator may not be able to precisely predict whether a proposal will pass, and legislators would not be able precisely predict the effects of a policy implemented in the current period. An argument can be made that such uncertainty exists in real-world legislative systems. We take it as evident, however, that uncertainty about the future is of a higher order of magnitude, and we bring out that contrast in the model by simply assuming complete information for the stage game in any given period. In any case, the addition of noise in the current period to the model would facilitate the equilibrium analysis. Put differently, we establish existence and various properties of equilibrium

[15] To see this, note that $u_i(x) + \theta_i \cdot x = -(x - (\hat{x}_i + \frac{1}{2}\theta_i)) \cdot (x - (\hat{x}_i + \frac{1}{2}\theta_i)) + \theta_i \cdot \hat{x} + \frac{1}{4}\theta_i \cdot \theta_i$, which is just the sum of a term, $\theta_i \cdot \hat{x} + \frac{1}{4}\theta_i \cdot \theta_i$, constant in x and the quadratic utility with ideal point $\hat{x}_i + \frac{1}{2}\theta_i$.

[16] We could, as well, elaborate the model by assuming an additional autocorrelated preference shock. What is needed is that there be some component of the preference shock that is independently distributed across periods.

without recourse to the assumption of noise in the current period.

Strategies and Payoffs A strategy in the game consists of two components giving the proposals of legislators when recognized to propose and the votes of legislators after a proposal is made. While these choices can conceivably depend on histories arbitrarily, we seek subgame perfect equilibria in which legislators use simple strategies. We are therefore interested in stationary Markov strategies, which we denote $\sigma_i = (\pi_i, \alpha_i)$. Our main focus will be on pure strategies, which, as we will show in the sequel, is without loss of generality. Thus, $\pi_i\colon \Re^d \times \Theta \to \Re^d$ is legislator i's proposal strategy, assumed measurable, where $\pi_i(q, \theta)$ is the policy proposed by i given status quo q and utility shocks θ; and $\alpha_i\colon \Re^d \times \Re^d \times \Theta \to \{0, 1\}$ is i's voting strategy, also assumed measurable, where $\alpha_i(y, q, \theta) = 1$ if i accepts proposal y given status quo q and utility shocks θ and $\alpha_i(y, q, \theta) = 0$ if i rejects. We let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a stationary strategy profile. We may equivalently represent voting strategies by the set of feasible proposals a legislator would vote for. We define this *acceptance set* for i as $A_i(q, \theta; \sigma) = \{y \in X \cup \{q\} : \alpha_i(y, q, \theta_i) = 1\}$. Letting C denote a coalition of legislators, we then define

$$A_C(q, \theta; \sigma) \;=\; \bigcap_{i \in C} A_i(q, \theta; \sigma) \quad \text{and} \quad A(q, \theta; \sigma) \;=\; \bigcup_{C \in \mathscr{D}} A_C(q, \theta; \sigma)$$

as the *coalitional acceptance set* for C and the *legislative acceptance set,* respectively. The latter consists of all policies that would receive the votes of all members of at least one decisive coalition, and would therefore pass if proposed.

Given strategies σ, we define legislator i's induced preferences in the game by

$$U_i(y, \theta_i; \sigma) = (1 - \delta_i)(u_i(y) + \theta_i \cdot y) + \delta_i v_i(y; \sigma),$$

where $v_i(x; \sigma)$ is i's continuation value at the beginning of period $t + 1$ from policy outcome x in period t.[17] We initially assume that legislators use "deferential" voting strategies, in the sense that when indifferent between a proposed policy and the status quo, they vote to accept. This assumption, which will turn out to be without loss of generality, then allows us to focus on no-delay equilibria, in which no legislator ever proposes a policy that is rejected. (In lieu of that, the legislator can just as well propose the status quo.) Our measurability assumptions on strategies imply that continuation values are also measurable, and therefore equilibrium continuation values satisfy

$$v_i(x; \sigma) = \int_q \int_\theta \sum_j p_j U_i(\pi_j(q, \theta), \theta_i; \sigma) f(\theta) g(q|x) d\theta dq \tag{1}$$

for all policies x.

To extend these ideas to allow for mixing and non-deferential voting, we let $\Pi_i\colon \Re^d \times \Theta \to \mathscr{P}(\Re^d)$ denote a mixed proposal strategy, where $\mathscr{P}(\Re^d)$ is the set of Borel probability measures on $\Re^d$. We equip this space with the weak* topology, and we assume Π_i is Borel measurable. Here, $\Pi_i(q, \theta)$ represents the distribution of i's policy proposal given status quo q and shocks θ. We then extend voting strategies to measurable mappings $\alpha_i\colon \Re^d \times \Re^d \times \Theta \to [0, 1]$, where $\alpha_i(q, \theta)$ is now the probability that i accepts proposal y given q and θ. A mixed strategy for legislator i is then $\Sigma_i = (\Pi_i, \alpha_i)$, and we let $\Sigma = (\Sigma_1, \ldots, \Sigma_n)$ denote a mixed strategy profile. Given a profile

[17] Note that these continuation values are "ex ante," in the sense that they are calculated at the beginning of the period, before q and θ are realized.

Σ of mixed strategies, we define induced preferences $U_i(y, \theta_i; \Sigma)$ as above, but the legislators' continuation values now have the following more complicated form:

$$v_i(x; \Sigma) = \int_q \int_\theta \sum_j p_j \int_y \big[\alpha(y, q, \theta; \Sigma) U_i(y, \theta_i; \Sigma) + (1 - \alpha(y, q, \theta; \Sigma)) U_i(q, \theta_i; \Sigma)\big] \Pi_i(q, \theta)(dy) f(\theta) g(q|x) d\theta dq, \tag{2}$$

where

$$\alpha(y, q, \theta; \Sigma) = \sum_{C \in \mathscr{D}} \left(\prod_{j \in C} \alpha_j(y, q, \theta) \right) \left(\prod_{j \notin C} (1 - \alpha_j(y, q, \theta)) \right)$$

is the probability that a proposal y is accepted by a majority of legislators. Note that, in (2), we now integrate over the policies proposed by each legislator i, and given a realization y from the mixed proposal strategy we now account for the possibility that y may pass with a probability intermediate between zero and one.

Legislative Equilibrium With this formalism established, we can now define classes of stationary Markov perfect equilibria of special interest. Intuitively, we require that legislators always propose optimally and that they always vote in their best interest. It is well-known that the latter requirement is ambiguous in simultaneous voting games, as arbitrary outcomes can be supported by Nash equilibria in which no voter is pivotal. We follow the standard approach of refining the set of Nash equilibria in voting subgames by requiring that legislators delete votes that are dominated in the stage game. Thus, we say a strategy profile σ is a *pure stationary legislative equilibrium* if the following conditions hold:

- for all shocks θ, every status quo q, and every legislator i, $\pi_i(q, \theta)$ solves

$$\max_y U_i(y, \theta; \sigma)$$
$$\text{s.t. } y \in A(q, \theta; \sigma),$$

- for all shocks θ, every status quo q, every proposal y, and every legislator i,

$$\alpha_i(y, q, \theta_i) = \begin{cases} 1 & \text{if } U_i(y, \theta_i; \sigma) \geq U_i(q, \theta_i; \sigma) \\ 0 & \text{else.} \end{cases}$$

This notion will be the main equilibrium concept of our analysis. Note that it not only imposes the requirement that legislators use pure strategies and eliminate stage-dominated voting strategies, but it also builds in the feature that voters defer to the proposer when indifferent. We then without loss of generality restrict proposers to the legislative acceptance set. Thus, this notion of equilibrium is relatively restrictive.

In contrast, we also define the following, relatively unrestrictive notion of equilibrium. We say a profile Σ of mixed strategies is a *mixed stationary legislative equilibrium* if

- for all shocks θ, every status quo q, and every legislator i, Π_i puts probability one on solutions to

$$\max_{y \in X \cup \{q\}} \alpha(y, q, \theta; \Sigma) U_i(y, \theta_i; \Sigma) + (1 - \alpha(y, q, \theta; \Sigma)) U_i(q, \theta_i; \Sigma),$$

- for all shocks θ, every status quo q, every proposal y, and every legislator i,

$$\alpha_i(y,q,\theta_i) = \begin{cases} 1 & \text{if } U_i(y,\theta_i;\Sigma) > U_i(q,\theta_i;\Sigma) \\ 0 & \text{if } U_i(y,\theta_i;\Sigma) < U_i(q,\theta_i;\Sigma). \end{cases}$$

The first difference between this notion of equilibrium and that of pure stationary legislative equilibrium is that we now allow a legislator, in case there are multiple optimal proposals, to mix over those proposals. The second difference is that we allow a legislator to vote with arbitrary probabilities when indifferent between a proposed policy and the status quo. Consistent with stage-game weak dominance, however, we require that legislators with strict preferences vote deterministically. This complicates the optimization problem of a proposer i, for the utility maximizing policies in the legislative acceptance set may no longer pass with probability one. Note that we can still, without loss of generality, restrict the proposer to the legislative acceptance set if we wish.

We say that a mixed strategy profile Σ is *equivalent* to a strategy profile σ if the policy outcome determined by (q,θ) is $\pi_i(q,\theta)$ with probability one. Formally, for all q, there exists a measure zero set $\Theta(q) \subseteq \Theta$ such that for all i and all $\theta \notin \Theta(q)$, we have: (i) if $\pi_i(q,\theta) \neq q$, then i proposes $\pi_i(q,\theta)$ and this passes with probability one, i.e., $\alpha(\pi_i(q,\theta),q,\theta;\Sigma)\Pi_i(q,\theta)(\{\pi_i(q,\theta)\}) = 1$, and (ii) if $\pi_i(q,\theta) = q$, then no proposal other than $\pi_i(q,\theta)$ passes with positive probability, i.e., $\int_{X\setminus\{q\}} \alpha(y,q,\theta;\Sigma)\Pi_i(q,\theta)(dy) = 0$. We separate the latter case from the first because there are two, payoff equivalent ways the status quo can prevail during a given period—it can be proposed and pass or a proposal can be rejected—a distinction that is irrelevant for the definition of equivalence. We will see that every mixed stationary legislative equilibrium is essentially pure, so that the added flexibility afforded by mixed strategies is moot, in equilibrium.

4 Main Results

In this section, we take up the existence and characterization of pure and mixed stationary legislative equilibria, robustness of equilibria, and the long run distribution of equilibrium policies.

Pure Stationary Legislative Equilibria The main result of this section is that there is a stationary legislative equilibrium satisfying a number of desirable technical properties.

Theorem 1 *There exists a pure stationary legislative equilibrium, σ, possessing the following properties.*

1. *Continuation values are differentiable: for every legislator i, $v_i(x;\sigma)$ is r-times continuously differentiable as a function of x.*
2. *Proposals are almost always strictly best: for every status quo q, almost all shocks θ, every legislator i, and every $y \in A(q,\theta;\sigma)$ distinct from the proposal $\pi_i(q,\theta)$, we have $U_i(\pi_i(q,\theta),\theta_i;\sigma) > U_i(y,\theta_i;\sigma)$.*
3. *Proposal strategies are almost always continuously differentiable: for every status quo q, almost all shocks θ, and every legislator i, $\pi_i(q,\theta)$ is continuously differentiable in an open set around (q,θ).*
4. *Constrained proposers almost always form minimal winning coalitions: for every status quo q, almost all shocks θ, and every legislator i, if $\pi_i(q,\theta) \neq q$ and there exists j such that $U_j(\pi_i(q,\theta),\theta_j;\sigma) = U_j(q,\theta_j;\sigma)$, then*

$$\{l \in N : U_l(\pi_i(q,\theta),\theta_l;\sigma) \geq U_l(q,\theta_l;\sigma)\} \setminus \{j\} \notin \mathscr{D}.$$

Part 1 of Theorem 1 establishes that equilibrium continuation values inherit the differentiable structure of the components of the model, u_i, h_ℓ, and g. By part 2, the equilibrium exhibited in Theorem 1 is, in a sense, strict: for almost all realizations of noise on preferences and the status quo, a proposer has a unique optimal policy choice. Part 3 of the theorem establishes a potentially useful technical property of equilibrium policy proposals: although equilibrium policy strategies will generally be discontinuous, they are differentiable on an open set of full measure.[18] In fact, in Lemma 3 of the appendix, we prove more. We show that for all q and almost all θ, there is a coalition C^* such that the proposer's optimization problem reduces to finding the best proposal subject to the assent of the members of C^*. Thus, the proposer's optimal proposal problem takes the standard form of a maximization problem subject to a finite number of equality and inequality constraints,

$$\begin{gathered}\max_y U_i(y,\theta_i;\sigma)\\ \text{s.t. } U_j(y,\theta_j;\sigma) \geq U_j(q,\theta_j;\sigma), j \in C^*\\ h_\ell(y) \geq 0, \ell \in K^{in}\\ h_\ell(y) = 0, \ell \in K^{eq},\end{gathered}$$

where we separate the "voting constraints" from the "feasibility constraints."

In Lemma 2 of the appendix we show that for all q and almost all θ, the well-known linear independence constraint qualification, or LICQ, holds at every policy distinct from the status quo that satisfies the voting and feasibility constraints.[19] With the observation in the preceding paragraph, this has the noteworthy implication that optimal proposals can be characterized by means of the Kuhn-Tucker first order conditions. In particular, the optimal proposal $\pi_i(q,\theta)$ is a critical point of the Lagrangian and the complementary slackness conditions hold: there exist λ_ℓ, $\ell \in K$, and λ_j, $j \in C^*$, such that

$$\begin{gathered}D_y U_i(\pi_i(q,\theta),\theta_i;\sigma) + \sum_{\ell\in K} \lambda_\ell D_y h_\ell(\pi_i(q,\theta)) + \sum_{j\in C^*} \lambda_j D_y U_j(\pi_i(q,\theta),\theta_j;\sigma) = 0\\ \lambda_j \geq 0 \text{ and } \lambda_j(U_j(\pi_i(q,\theta),\theta_j;\sigma) - U_j(q,\theta_j;\sigma)) = 0,\ j \in C^*\\ \lambda_\ell \geq 0 \text{ and } \lambda_\ell h_\ell(\pi_i(q,\theta)) = 0,\ \ell \in K^{in}\\ h_\ell(\pi_i(q,\theta)) = 0,\ \ell \in K^{eq}.\end{gathered}$$

Furthermore, we show that the complementary slackness conditions almost always hold strictly,[20] that the second order sufficient conditions for a constrained maximizer almost always hold, and that optimal proposals are almost always strongly stable in the sense of Kojima (1980).[21]

Part 4 of Theorem 1 establishes conditions under which a proposer will form minimal winning coalitions. We show that for all q and almost all θ, if the proposer is "constrained," in the sense that the optimal policy proposal renders at least one other legislator indifferent between the proposal and the status quo, then all legislators who are indifferent between the proposal and the status quo are necessary coalition partners: the proposal fails if we remove any such legislator's assent. An implication is that if the voting rule is a quota rule, then the assent of all legislators approving

[18]The exact form of the derivative is calculated by the implicit function theorem. See equations (5.1)–(5.3) in Fiacco and Ishizuka (1990).

[19]A policy y satisfies LICQ if the gradients of the binding voting and feasibility constraints are linearly independent.

[20]That is, the multipliers corresponding to binding inequality constraints are strictly positive.

[21]See our appendix, preceding Lemma 2, for the definition of strong stability.

the proposal in these situations is necessary for the proposal to pass, and the winning coalition is of minimum size. This is reminiscent of Riker's (1962) size principle, which maintains that winning coalitions are of minimal size necessary in order for a proposal to pass, and no larger. Part 4 can be viewed as a formalization of the size principle in a general, non-cooperative, dynamic model of policy making. In fact, the theorem suggests a caveat to the size principle: if a decisive coalition of legislators prefer the proposer's ideal feasible point to the status quo, so the proposer is "unconstrained," then there is nothing in the logic of Theorem 1 necessitating that the proposal receives only the support of a minimum winning coalition.

As expected, the proof of Theorem 1 proceeds by defining a suitable mapping, establishing the existence of a fixed point, and then verifying that it corresponds to a stationary legislative equilibrium with the claimed properties. Here, we sketch some key steps in the proof of existence. Let $C^r(\Re^d, \Re^n)$ denote the space of r-times continuously differentiable mappings from $\Re^d$ to $\Re^n$, endowed with the topology of C^r-uniform convergence on compacta.[22] Given a vector $v = (v_1, \dots, v_n) \in C^r(\Re^d, \Re^n)$ of continuation value functions, define $U_i(y, \theta; v)$ and $A_i(q, \theta; v)$, in the obvious way, as the induced utilities and acceptance sets when continuation values are given by v. We then consider a legislator i's optimal proposal problem,

$$\begin{aligned} &\max_y U_i(y, \theta_i; v) \\ &\text{s.t. } y \in A(q, \theta; v), \end{aligned} \tag{3}$$

and we let $\pi_i(q, \theta; v)$ denote a selection from the solutions to this program. This selection then determines a new vector of continuation values, $\hat{v} = (\hat{v}_1, \dots, \hat{v}_n)$, for the legislators, and we define ψ as the mapping that takes the vector v to the new vector $\hat{v}$, i.e., $\psi(v) = \hat{v}$.

The existence proof consists in verifying that ψ satisfies the conditions of Glicksberg's fixed point theorem. The technical role of the noise on the status quo, is to allow us to restrict the domain and range of ψ to a compact subset of $C^r(\Re^d, \Re^n)$. To see how, note that the new continuation value $\hat{v}_i$ of legislator i is defined by

$$\hat{v}_i(x) = \int_q \int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\theta dq, \tag{4}$$

and note further that the current period's policy choice x enters this continuation value only through the density $g(q|x)$. Thus, ψ is, in essence, the convolution of $\int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) d\theta$, a generally discontinuous function of q, with the function $g(q|x)$. This allows us to smooth the legislators' payoffs and to restrict the domain and range of ψ to a compact set. Indeed, assuming for the sake of argument that $r = \infty$, we define $\mathscr{V}$ to consist of all functions $v \in C^\infty(\Re^d, \Re^n)$ such that each v_i is bounded in absolute value by c and derivatives of v of all orders are bounded in norm by $\sqrt{n}abc$. This is a compact space, and it is straightforward to verify that ψ maps $\mathscr{V}$ into itself.

The preference shock plays a critical role in establishing continuity of the mapping ψ. The argument relies on the result, underlying part 2 of Theorem 1, that for any given v, for every status quo q, and almost all shocks θ, the proposer's maximization problem has a unique solution. Thus, the selection $\pi_i(q, \theta; v)$ is uniquely pinned down almost everywhere. The intuition behind this uniqueness result is straightforward: if legislator i is indifferent between proposing two policies

[22] A sequence $\{\phi^m\}$ of functions converges to ϕ in this topology if and only if for every compact set $Y \subseteq \Re^d$ and all derivatives of orders $0, 1, \dots, r$ of $\phi^m - \phi$ converge uniformly in norm to zero on Y. See Mas-Colell (1985).

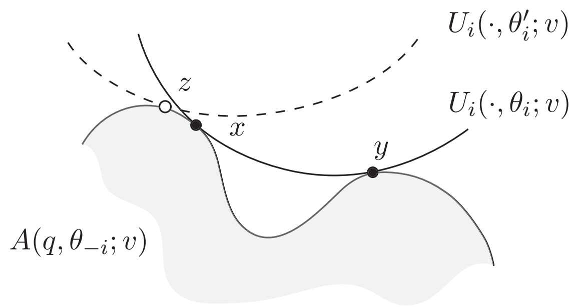

Figure 1: Genericity of unique maximizer

for one realization of θ_i, then, generically, a perturbation θ_i' of θ_i will break that indifference. This is depicted in Figure 1, where policies x and y maximize $U_i(\cdot, \theta_i; v)$ over $A(q, \theta; v)$, shaded in the figure. Note that the constraint in i's optimal proposal problem in (3) can be reformulated to exclude the constraint requiring that i accept his own proposal, so we can write the constraint set as $A(q, \theta_{-i}; v)$, independent of θ_i. Then a small perturbation to θ_i' leads to a unique maximizer z. Key here is the fact that a perturbation of θ_i does not affect the payoffs of other legislators or, therefore, the constraints of i's maximization problem.

Having proved uniqueness of the selection $\pi_i(q, \theta; v)$ almost everywhere, the preference shock delivers continuity of the mapping ψ as follows. Using differentiability of the $U_i(y, \theta_i; v)$, we apply the transversality theorem to deduce that for any given v, for every status quo q, almost all shocks θ, and every policy $y \in A(q, \theta; v)$ distinct from q, LICQ holds. This is depicted in Figure 2. Here, for simplicity, we suppose the legislative acceptance set is the intersection of legislators 1's and 2's acceptance sets, which are shaded. Although the gradients of legislators 1 and 2 are linearly dependent at y, i.e., LICQ is violated, a small shock to θ_1 will lead to a perturbation of the acceptance set of legislator 1, given by the dashed curve in the figure. We then have the generic situation, in which LICQ is satisfied over the legislative acceptance set. This in turn implies lower hemicontinuity of the legislative acceptance set correspondence $A(q, \theta; v)$ for almost all θ, and by the theorem of the maximum, a legislator's optimal proposal $\pi_i(q, \theta; v)$ will be jointly continuous in $(q, \theta; v)$ for almost all θ.[23] This implies that the integrand in (4),

$$\int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\theta,$$

is continuous as a function of (x, v), and continuity of $\hat{v}_i(x) = \psi(v)(x)$ in (x, v) follows by an application of Lebesgue's dominated convergence theorem. It is then straightforward to apply this argument to higher derivatives, delivering continuity of the mapping ψ in the topology of C^r-uniform convergence on compacta, thereby permitting the application of Glicksberg's theorem.

Mixed Stationary Legislative Equilibria The next result justifies our focus on pure stationary legislative equilibria. It establishes that every mixed equilibrium is equivalent to a pure

[23] In the appendix, we give a more powerful, though more involved, argument that the optimal proposal is in fact differentiable.

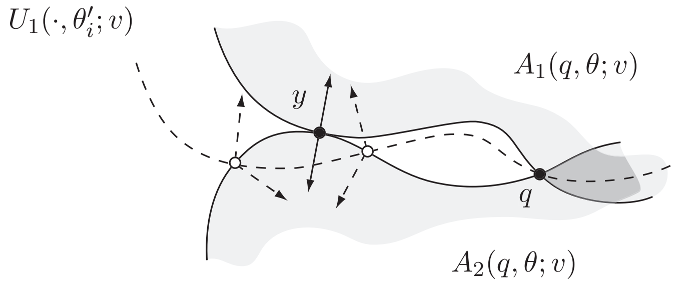

Figure 2: Genericity of LICQ

one. Furthermore, because every pure equilibrium is a special case of mixed, it shows that every pure stationary legislative equilibrium satisfies the properties of Theorem 1.

Theorem 2 *Every mixed stationary legislative equilibrium is equivalent to a pure stationary legislative equilibrium that satisfies the properties in parts 1–4 of Theorem 1.*

Much of the intuition for this result has already been discussed. Given a mixed stationary legislative equilibrium, with continuation value v, our earlier observation that the solution, $\pi_i(q,\theta)$, to a legislator's optimal proposal problem in (3) is almost always unique carries over without change. This does not immediately rule out the possibility of non-degenerate mixed strategies, however, because one or more legislators may be indifferent between $\pi_i(q,\theta)$ and the status quo, and these legislators could conceivably vote to accept with probability less than one. But our subsequent claim that LICQ holds at every policy $y \in A(q,\theta;v)$ distinct from q relied only on the differentiability of the equilibrium continuation values v, and inspection of (2) reveals that even in a mixed equilibrium, continuation values will inherit the differentiability assumed in the model: the current period's policy x enters the righthand side of (2) only through the function $g(q|x)$, which is appropriately smooth in x. An implication is that the proposer can find policies arbitrarily close to $\pi_i(q,\theta)$ that are strictly better than the status quo for a decisive coalition of legislators. Such proposals will pass with probability one, and existence of an optimal proposal (implied by equilibrium) demands that $\pi_i(q,\theta)$ will also pass with probability one. Since $\pi_i(q,\theta)$ is the unique solution to (3), any optimal mixed proposal strategy must put probability one on that policy.

Continuity of the Equilibrium Correspondence A desirable property of equilibria is robustness, which we formalize in terms of upper hemicontinuity of the equilibrium correspondence with respect to the parameters of the model. In our framework, a model is specified by parameters $\gamma = ((p_i,u_i)_{i\in N},\delta_i,X,f,g)$, where X is a compact set defined by a finite number of equality and inequality constraints such that the gradients of binding constraints are linearly independent. Let Γ denote a metric space of possible parameterizations, where we assume that each satisfies our maintained assumptions from Section 3. In particular, we assume that there exist a compact set $\tilde{X}$ (with Lebesgue measure a) and bounds b and c such that for all $\gamma \in \Gamma$, (i) the function $|u_i^\gamma(x) + \theta_i \cdot x| f^\gamma(\theta)$ is bounded by c over $i \in N$, $\theta \in \Theta$, and $x \in \tilde{X}$, (ii) for all x, the support of $g(\cdot|x)$ is contained in $\tilde{X}$, and (iii) for almost all q, $g^\gamma(q|x)$ is r-times continuously differentiable in x, (iv) if $r < \infty$, then derivatives of order $1,\ldots,r$ are bounded in norm by b and the r-th derivative is Lipschitz continuous with modulus b, and if $r = \infty$, then derivatives of all orders $1,2,\ldots$ are bounded in norm by b. In addition, we assume that the parameterization is continuous: (v) p_i^γ and δ_i^γ are continuous in γ, (vi) $u_i^\gamma(x)$ is jointly continuous in (x, γ), (vii) X^γ is continuous in γ with the Hausdorff metric on closed subsets of $\Re^d$, (viii) for all θ, $f^\gamma(\theta)$ is continuous in γ, and (ix) for all q, $g^\gamma(q|x)$ is continuous in (x, γ). Note that our parameterization is especially general with respect to the set of feasible policies, for we do not assume that policy spaces are generated by a common set of parameterized feasibility constraints.

We define the equilibrium correspondence $E\colon \Gamma \rightrightarrows C^r(\Re^d, \Re^n)$ so that $E(\gamma)$ consists of the set of pure stationary legislative equilibrium continuation values $v \in C^r(\Re^d, \Re^n)$. Theorem 1 shows that E is nonempty-valued. The next result establishes that the equilibrium correspondence E is upper hemicontinuous. Thus, equilibria are robust in the sense that a small perturbation of the parameters of our model cannot produce new equilibria far from the original equilibrium set.

Theorem 3 *The equilibrium correspondence $E\colon \Gamma \rightrightarrows C^r(\Re^d, \Re^n)$ is upper hemicontinuous.*

The proof of upper hemicontinuity shares much of the structure of the argument sketched above for existence in Theorem 1. We note there that for a fixed v, for all q, and almost all θ, the legislators' optimal proposals, $\pi_i(q, \theta; v)$, are continuous at (q, θ), and the same observation holds when we allow the parameters of the model to vary as described in (i)–(ix). Specifically, for all v, all q, and almost all θ, π_i is jointly continuous in (q, θ, v). Once this is proven, upper hemicontinuity of the equilibrium correspondence follows from the application of Lebesgue's dominated convergence theorem. Theorem 3 is stated in terms of continuation value functions because our topology on this function space allows for a compact statement in terms of upper hemicontinuity, but we could have as well stated a type of upper hemicontinuity result in terms of proposal strategies: if $\gamma^m \to \gamma$, if $v^m \to v$, and if $\sigma^m = (\pi^m, \alpha^m)$ is a stationary legislative equilibrium in each γ^m, then there is a stationary legislative equilibrium $\sigma = (\pi, \alpha)$ in γ such that for all i, all q, and almost all θ, $\pi_i^m(q, \theta) \to \pi_i(q, \theta)$.

Ergodic Properties of Stationary Legislative Equilibria A stationary legislative equilibrium, say σ^*, determines a stochastic process on policies, and we may then consider the equilibrium dynamics of policy outcomes in our model. Given Borel measurable $Y \subseteq \Re^d$, let I_Y denote the indicator function of Y. We define the transition probability on policy outcomes by

$$P(x, Y) = \int_q \int_\theta \sum_{i \in N} p_i I_Y(\pi_i^*(q, \theta)) f(\theta) g(q|x) d\theta dq,$$

which is the probability, conditional on policy outcome x this period, of a policy outcome in the set Y next period. We define the associated Markov operator T on the space of bounded, Borel measurable functions $\phi\colon X \to \Re$ by $T\phi(x) = \int \phi(z) P(x, dz)$. The adjoint T^* operates on the Borel measures on X, denoted μ, and is defined by $T^*\mu(Y) = \int P(x, Y)\mu(dx)$. This describes the distribution of outcomes in the next period, given a distribution μ of policy outcomes in the current period. The iterates of T^*, denoted T^{*m}, give the distribution of policy outcomes m periods hence and are therefore key in describing the long run policy outcomes of the model.

It is straightforward to show that T maps continuous functions to continuous functions and, therefore, satisfies the Feller property. It is also tight, and it therefore immediately admits at least one invariant distribution μ^*, such that $\mu^* = T^*\mu^*$. Thus, each stationary legislative equilibrium determines an "ergodic Markov equilibrium," in the sense of Duffie et al. (1994). Furthermore, we can show that under the weak assumption that the density g is bounded, P satisfies Doeblin's condition (see Futia (1982), Definition 4.8), so from any initial distribution μ on X, the sequence of long run average distributions, $\frac{1}{m}\sum_{t=1}^m T^{*t}\mu$, $m = 1, 2, \ldots$, converges to an invariant distribution in the total variation norm. This result is similar in spirit to that of Hellwig (1980), who uses Doeblin's condition to establish ergodic properties of temporary equilibria. While it provides a minimal characterization of long run policy outcomes, however, the result is weak in several respects: it concerns the long run average distributions, rather than the distribution of policy outcomes in each period t; the limiting invariant distribution can depend on the initial distribution; and the rate of convergence is only known to be arithmetic.

Under further restrictions on the transition probability, standard results on Markov processes can be applied to address these shortcomings. While the transition probability in our model is endogenous and a full characterization of equilibrium strategies is beyond the scope of this analysis, we can guarantee the desired properties of the transition probability through restrictions on the exogenous density $g(q|x)$. Specifically, we assume there is a status quo q_0 that belongs to the support of $g(\cdot|x)$ for every choice of policy. Though this assumption is not entirely unrestrictive, we allow for the status quo densities to place arbitrarily low (but positive) probability near q_0, so it should not pose an impediment to applications of the model. With this assumption, we obtain convergence of per period policy distributions instead of the long-term average distribution, we deduce uniqueness of the invariant distribution, and we obtain geometric convergence in the total variation norm. Thus, from any starting point, each stationary legislative equilibrium of our model generates unambiguous predictions of long run equilibrium policy outcomes.

Theorem 4 *Let σ^* be a stationary legislative equilibrium and T the associated Markov operator.*

1. *The Markov operator T admits at least one invariant distribution.*
2. *If g is bounded on $\tilde{X} \times \tilde{X}$, then given any initial distribution μ, the sequence of long run average distributions, $\frac{1}{m}\sum_{t=1}^{m} T^{*t}\mu$, converges arithmetically to an invariant distribution μ^* of T^* in the total variation norm: there is a constant $M > 0$ such that*
$$\left|\left|\sum_{t=1}^{m} T^{*t}\mu - \mu^*\right|\right| \leq \frac{M}{m}$$
for all m.
3. *If, in addition, there exists $q_0 \in \tilde{X}$ that is contained in the support of $g(\cdot|x)$ for all $x \in \tilde{X}$, then T^* admits a unique invariant distribution, say μ^*. Given any initial distribution μ, the sequence of iterates, $T^{*m}\mu$, converges geometrically to μ^* in the total variation norm: there are constants $M, \epsilon > 0$ such that*
$$||T^{*m}\mu - \mu^*|| \leq \frac{M}{(1+\epsilon)^m}$$
for all m.

While part 1 of this result falls out of the structure of our framework and part 2 holds very generally, the important strengthening in part 3 relies on the regularity properties of stationary legislative equilibria established in Theorem 2. In particular, it uses the fact that in every stationary legislative equilibrium, the proposal strategies of the legislators are differentiable for almost all q and θ (in fact, we use only continuity). Nevertheless, Theorem 4 provides only rather gross sufficient conditions that do not take advantage of the details of equilibrium strategies. We conjecture that in more structured environments, uniqueness of the ergodic distribution may follow under even weaker conditions on the status quo density.

5 Numerical Example

In this section, we supplement our theoretical analysis with an investigation of a simple example featuring five legislators, an interval of feasible policies, and quadratic stage utilities. Despite its simplicity, the example highlights the strategic incentives of legislators in our dynamic framework and reveals perhaps unexpected complexities of equilibrium behavior. We do not solve for an equilibrium of this example by analytical methods. Instead, we employ numerical methods to compute an equilibrium, which we anticipate would be the approach used in applications, certainly in more elaborate specifications of the model. In view of the importance of numerical techniques for applications, we devote part of this section to a discussion of computational issues, taking an initial step toward the development of general numerical methods for the approximation of stationary legislative equilibria. In particular, in the rest of this section we first describe the example, we then outline an algorithm for the numerical solution of an arbitrary specification of the model, and finally we return to the example to present the results of our numerical investigation.

We consider a legislature with $n = 5$ members, each of whom has negative quadratic stage preferences $u_i(x) = -(x - \hat{x}_i)^2$, $i = 1, \ldots, 5$ over a one-dimensional continuum, $X = [0, 4]$. A policy in X may represent, e.g., a level of taxation in some appropriate tax scale. We set the ideal points of the five legislators at $\hat{x}_1 = 1$, $\hat{x}_2 = 1.5$, $\hat{x}_3 = 2$, $\hat{x}_4 = 2.8$, and $\hat{x}_5 = 3$. Thus, in terms of the underlying stage preferences, legislator 3 is the median with an ideal point at 2, with the two legislators located to the right of the median having ideal points closer to each other and further away from the median than the ideal points of legislators 1 and 2 located to the left of the median. We will see that this apparently minor asymmetry has noticeable repercussions for equilibrium behavior. Assume a common discount factor $\delta_i = 0.9$, and assume equal recognition probabilities $p_i = \frac{1}{5}$, for all $i = 1, \ldots, 5$. We assume that preference shocks θ_i are independently distributed, so that each θ_i is uniform with support in $[-0.1, 0.1]$, and we assume that given policy choice x in period t, the status quo in period $t + 1$ is drawn from $g(\cdot|x)$, which is the density of the Beta distribution with shape parameters $\hat{\alpha} = \hat{\beta} = 4$ and support equal to $[\hat{\mu}(x) - 0.1, \hat{\mu}(x) + 0.1]$, where $\hat{\mu}(x) = 0.1 + .95x$. This specification satisfies the differentiability conditions of Section 3, and we may set $\tilde{X} = X$, as the status quo belongs to X in all periods.

In general, given a model with a continuum X of feasible policies, we consider an algorithm for computing stationary legislative equilibria by means of an increasing sequence of finite grids on the policy space. To be concrete, let $\{X^m\}$ be a sequence of finite approximations to X. For each m, define a corresponding "quasi-discrete" model that is otherwise identical to the original model except for the fact that feasible proposals (but not status quo, q) are now constrained to lie in X^m. The quasi-discrete model is a special case of the our legislative model model, for as mentioned above, we can obtain the finite set X^m of feasible policies by appropriately specifying equality constraints h^m_ℓ, $\ell \in K^{eq,m}$. Therefore, Theorem 1 yields at least one pure stationary legislative equilibrium, with continuation value function v^m, in each quasi-discrete model. Importantly, our next result then establishes that the sequence $\{v^m\}$ admits a convergent subsequence, and that the limit of any such subsequence is an equilibrium continuation of the continuum model.

Theorem 5 *Given $\gamma = ((p_i, u_i, \delta_i)_{i \in N}, X, f, g) \in \Gamma$, let $\{\gamma^m\}$ be a sequence of quasi-discrete models such that $\gamma^m \to \gamma$ and $\gamma^m = ((p_i, u_i, \delta_i)_{i \in N}, X^m, f, g) \in \Gamma$ for all m.*

1. *For all m, there exists $v^m \in E(\gamma^m)$.*
2. *For every such sequence $\{v^m\}$ of equilibrium continuation values, there exists a convergent subsequence with limit $v \in E(\gamma)$.*

$$
\begin{array}{cccccccc}
\text{Grids} & & & & & & & \begin{array}{c}\text{Eq. cont.}\\ \text{values}\end{array} \\
X^1 & & \hat{v}^{1,0} & \hat{v}^{1,1} & \hat{v}^{1,2} & \cdots & \rightarrow & \hat{v}^1 \\
X^2 & \hat{v}^1 = & \hat{v}^{2,0} & \hat{v}^{2,1} & \hat{v}^{2,2} & \cdots & \rightarrow & \hat{v}^2 \\
X^3 & \hat{v}^2 = & \hat{v}^{3,0} & \hat{v}^{3,1} & \hat{v}^{3,2} & \cdots & \rightarrow & \hat{v}^3 \\
\downarrow & & \vdots & \vdots & \vdots & & & \downarrow \\
X & & & & & & & \hat{v}^*
\end{array}
$$

Figure 3: A Schematic Representation of Computation Algorithm

Theorem 5 provides a starting point for our computational approach, as it suggests an iterative algorithm for computing equilibria based on computing equilibria of quasi-discrete versions of the model. The algorithm proceeds in a number of steps, $m = 1, 2, \ldots$, which we describe with reference to an arbitrary step m. We specify a quasi-discrete version of the model with a finite grid $X^m \subseteq X$ of feasible policies. By Theorem 1, this model admits at least one pure stationary legislative equilibrium. In order to approximate the continuation value functions of an equilibrium of this quasi-discrete model, we begin with an initial guess $\hat{v}^{m,0}$ (in step $m = 1$, this guess may be arbitrary). Given this initial guess, we can trivially compute best response proposal strategies, which we denote by $\pi_i^{m,0}(q,\theta)$, via a grid search over the $|X^m|$ (or possibly $|X^m| + 1$, including the status quo) candidates for optimal proposal. A new set of best response continuation value functions, $v^{m,1}$, are given by the integral

$$
v_i^{m,1}(x) = \int_q \int_\theta \sum_{j \in N} p_j \Big[(1-\delta_i)(u_i(\pi_j^{m,0}(q,\theta)) + \theta_i \cdot \pi_j^{m,0}(q,\theta)) + \delta_i \hat{v}_i^{m,0}(\pi_j^{m,0}(q,\theta)) \Big] f(\theta) g(q|x) d\theta dq
$$

for each x and i, resulting in a vector $v^{m,1} = (v_1^{m,1}, \ldots, v_n^{m,1})$. We compute the value of $v_i^{m,1}(x)$ using numerical integration at a finite number of interpolation points $\{x_1, ..., x_\kappa\}$, and we use these interpolation pairs $(x_\iota, v_i^{m,1}(x_\iota))$, $\iota = 1, \ldots, \kappa$, to obtain an approximation of $v^{m,1}$, denoted $\hat{v}^{m,1}$, by Chebyshev interpolation. Next, we perform another cycle of the above steps in order to calculate approximate best response continuation values to $\hat{v}^{m,1}$, and we continue iteratively, generating a sequence $\hat{v}^{m,1}, \hat{v}^{m,2}, \hat{v}^{m,3}, \ldots$, until this sequence converges to some $\hat{v}^m$.

A limit, $\hat{v}^m$, obtained by this process approximates an equilibrium continuation value of the quasi-discrete model with feasible set X^m. We then proceed with the algorithm to a finer grid X^{m+1}, using $\hat{v}^m$ as the initial set of continuation values, i.e., $\hat{v}^{m+1,0} = \hat{v}^m$. The algorithm generates a sequence $\hat{v}^{m+1,1}, \hat{v}^{m+1,2}, \hat{v}^{m+1,3}, \ldots$ of continuation values and, upon convergence, yields $\hat{v}^{m+1}$. This in turn approximates an equilibrium of the quasi-discrete model with feasible set X^{m+1}. We then increase the grid size to X^{m+2}, now setting $\hat{v}^{m+2,0} = \hat{v}^{m+1}$ as the initial value, and so on. This algorithm generates a sequence $\hat{v}^1, \hat{v}^2, \ldots$ of (approximate) equilibria of the quasi-discrete models, and by part 2 of Theorem 5 (assuming our approximations $\{\hat{v}^m\}$ are sufficiently close) this sequence has at least one accumulation point, say, $\hat{v}^*$, which corresponds to an equilibrium of the continuum model. A schematic representation of this algorithm is depicted in Figure 3.

Before we proceed to report on the results from the implementation of this algorithm in the context of our example, we offer a number of remarks. While Theorem 5 ensures the ("vertical") convergence of a subsequence of equilibrium continuation values from the quasi-discrete models, there is no guarantee of ("horizontal") convergence to an equilibrium continuation value for any fixed grid size. This underlines the importance of the initial guess for a given grid size, as we expect the prospects for, and speed of, convergence will depend on the proximity of the initial value $v^{m,0}$ to true equilibrium continuation values. A virtue of our algorithm is that if we do get convergence for one grid size, then this provides (at relatively low cost) a good initial guess for the computation of equilibrium continuation values at the next grid size. Furthermore, by construction, the equilibrium generated by the algorithm is robust, in a sense, to discretizations of the set of feasible policies, since the algorithm stops when additional changes in grid size no longer generate appreciable improvements on computed equilibria. An additional advantage of using a sequence of quasi-discrete models is that the set of possible proposals is a finite set, so that we are guaranteed exact computation of optimal proposals at each iteration of the algorithm. This is important as there is no guarantee, in general, that the optimization problem (3) of a proposer is well behaved: even if the policy space is convex and stage utilities are concave, we have no theoretical grounds to believe that the objective function of the proposer must be quasi-concave or that the legislative acceptance set is convex. Furthermore, both the objective function as well as the acceptance set change as computation iterates over a sequence of continuation values. Thus, while numerical optimization over a grid is computationally costly, it is a safe alternative to a potentially very hard optimization problem in the continuum model.

We implemented the numerical algorithm in our example with a convergence criterion of 10^{-5}, stopping the computation for a fixed grid size when the distance between the Chebyshev coefficients of successive continuation values is smaller than 10^{-5}. We started with an initial grid size of 4000, increasing the grid size by 50% in each grid iteration. We used the same convergence criterion of a distance smaller than 10^{-5} between the coefficients of successive iterates $\hat{v}^m, \hat{v}^{m+1}$ for different grid sizes. We achieved overall convergence after six grid iterations, at a grid size of 30,375. Even though these computations can become expensive in more demanding applications, we are optimistic about the possibility of achieving gains in efficiency by using modified computation schemes. We report computed proposal strategies $\hat{\pi}_i^*$, expected payoffs $\hat{U}_i^*$, and continuation values $\hat{v}_i^*$, for each of the five legislators in Figure 4.

The first row of plots in Figure 4 represents proposal strategies $\hat{\pi}_i^*$. The solid lines represent proposals for the case $\theta = 0$, while the dotted lines reflect proposals for the maximum and minimum possible θ_i, holding $\theta_{-i} = 0$. It is evident from these proposal strategies that proposers to the left of the median compromise (relative to their stage utility ideal points) by proposing policies closer to the ideal point of the median, located at 2. This is also clear by the second row of plots, which represent expected payoffs $\hat{U}_i^*$. These are single-peaked but (with the exception of the median) neither symmetric nor concave, in contrast to the quadratic stage preferences of the legislators. Furthermore, legislators 1 and 2, with stage ideal points at 1 and 1.5, have equilibrium preferences with induced ideal points significantly to the right of their stage ideal points, closer to the ideal point of the median, at 2. An interesting feature of the equilibrium, is the asymmetry in strategic behavior between legislators located at the left and the right of the median. Unlike legislators 1 and 2, legislators 4 and 5 have induced ideal points that are much closer to their stage ideal points; indeed, they are near 2.8 and 3. This difference is also manifest in the continuation value plots for these legislators, displayed in the third row of Figure 4. It is evident from these functions that these legislators expect the highest future payoffs from policies that are at *both* extremes of the policy space. This is because legislators 1 and 2 exhibit the compromising behavior we have already discussed. As a result, legislators 4 and 5, who are more cohesive, having closer stage preferences, face good future prospects with extreme policies. Such policies allow them to implement their ideal points if they propose, while the other legislators propose moderate policies anyway. On the other

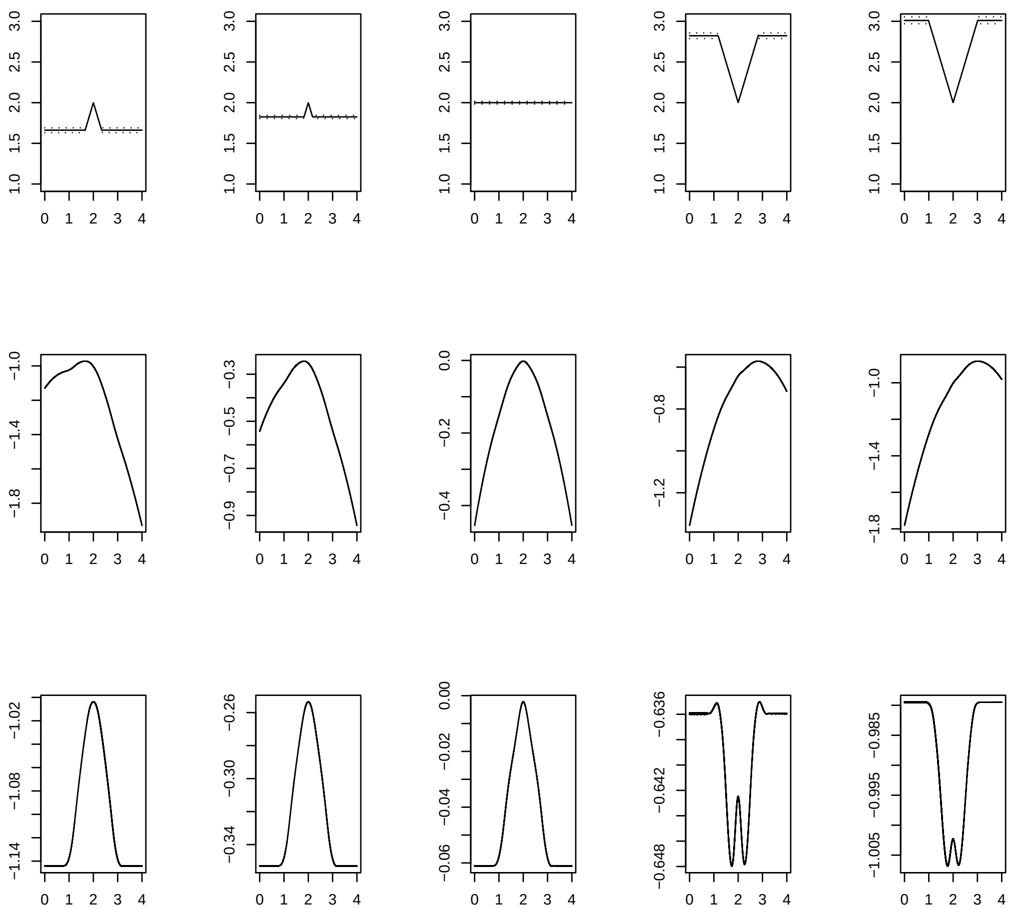

Figure 4: Proposal Strategies, Expected Payoffs, and Continuation Value Functions for 5-legislator Example

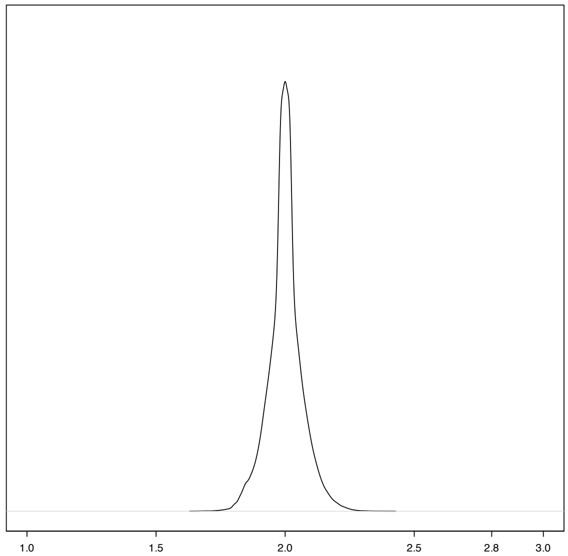

Figure 5: Ergodic Distribution in Computed Equilibrium of the Example

hand, legislators 1 and 2 have different incentives, since they expect extreme policies by legislators 4 and 5 when the status quo is extreme.

We conclude this section by reporting, in Figure 5, the ergodic distribution of the Markov process over policies induced by the computed equilibrium, which we compute using Markov Chain Monte Carlo simulation. Although the uniqueness condition of Theorem 5 does not hold in our example, it is straightforward to infer from the nature of the legislators' proposal strategies that the equilibrium induces a unique invariant distribution. This distribution has most of its mass concentrated at the median, with some asymmetry evident as a consequence of the location of the equilibrium ideal points of the other four legislators. Thus, due to the noise in the status quo and legislators' preference shocks, our model implies a distribution over future policies in the dynamic setting we consider, even in a one-dimensional policy space in which a median voter is well-defined. This distribution seems plausible in real societies, exactly because we expect that participants' preferences and status quo policies are subjected to such shocks over time. Of course, we expect that this distribution becomes more concentrated around the median as the support around the noise elements in the model become smaller.

6 Discussion

We establish existence of stationary Markov perfect equilibria satisfying a number of desirable regularity properties in a general model of legislative policy making. We impose no constraints on the dimensionality of the policy space, we do not assume convexity conditions on policy preferences, and we allow for any voting rule that can be expressed in terms of a collection of decisive coalitions. The main technical assumption we impose is differentiability, which, in combination with uncertainty about future policy preferences and noise in the implementation of future policies, allows us to bring methods of differentiable topology to bear on the existence problem. For reasons of space, we have limited the scope of our analysis to a benchmark model that is austere, in the sense that we abstract away from much of the detail of real-world political institutions. The model encapsulates all of the difficult technical issue we would encounter in more complex models while offering advantages of efficiency in presentation. But our approach to existence and related issues is very general and extends to a much larger class of models that can capture a substantial amount of institutional detail. Here, we discuss some of the structure we are able to incorporate.

We first observe that our analysis extends immediately to finite-horizon versions of the legislative model. Our equilibrium construction centers on a best response continuation value mapping ψ, the fixed points of which correspond to stationary legislative equilibria. In the finite-horizon model, we may simply iterate this mapping to generate subgame perfect equilibrium continuation values: letting v^0 be the profile of zero functions, equilibrium continuation values following a policy choice in the second-to-last period are just $v^1 = \psi(v^0)$. Equilibrium continuation values following the third-to-last period are $v^2 = \psi(v^1)$, and so on. By induction, continuation values following a policy choice in the first period of the T-period version of the model are given by $v^{T-1} = \psi(v^{T-2})$. What is more, the arguments of the appendix (specifically, part 3 of Lemma 1) establish that these equilibrium continuation values are unique (up to sets of measure zero) among all mixed strategy subgame perfect equilibria. Thus, the finite-horizon legislative model is relatively tractable, though equilibria generated in this way will not be stationary in the strong sense of the equilibria obtained in Theorem 1 for the infinite-horizon model.

To accommodate more complex structure, we can augment the model with a Euclidean space of states s, and the parameters of the model—stage utilities, discount factors, the feasible policies, the voting rule, the identity of the proposer, and the status quo—can depend on s quite arbitrarily. Given current state s and policy outcome x, the next period's state s' and preference shock θ' can be drawn from a density $\phi(s', \theta'|x, s)$ with appropriate bounds on derivatives, permitting dependence on the endogenous policy choice variable x. This flexibility allows us to model the fine details of real-world legislative and parliamentary processes, such as the committee system in the US House and Senate. Because the feasible set, the voting rule, and the proposer are functions of the state, we model a committee system by varying the voting rule and the set of feasible policies with the proposer in such a way that feasible policies are restricted to the policy jurisdiction of the committee to which the proposer belongs, and so that the assent of the committee, along with the that of the floor, is required for passage. Control over such details opens the opportunity for a relatively fine-tuned analysis of constitutional design issues. This richer version of the model also allows us to encode elections within the exogenous state variable, which can determine the composition of the legislature in any period and permits the analysis of electoral incentives on policy-making.

Though further from the framework of the current paper, our approach also extends to models in which an electorate is introduced explicitly and endogenous elections are held at regular intervals over time. In this context, however, our approach does not deliver full stationarity, for in order to address convexity issues we must allow voters to vote retrospectively, i.e., condition on the policy outcome in the period prior to the election. We can also consider protocols more complex than the standard take-it-or-leave-it offer framework common in the literature on non-cooperative bargaining and legislative policy-making. We conjecture that our existence result carries over to a version of the model in which an initial proposal by one legislator is pitted against an amendment by another, with a final vote between the winner and the status quo. This "amendment agenda" structure more closely resembles the procedure under an open rule in the US House and Senate. Pursuing this line of reasoning further, we conjecture that we can augment our stage game with an arbitrary finite extensive form game of perfect information, which allows us to incorporate moves by key players in the legislative game, such as the assignment of legislators to committees by party leaders.

The success of these applications depends on the development of fruitful, probably numerical, techniques for the analysis of the model. We have implemented one algorithm for computation of equilibria in the context of a relatively simple model with five legislators determining policy in a one-dimensional space. It is conceptually straightforward to extend the algorithm to more legislators and more dimensions, but we leave for future research the issue of developing cost-efficient algorithms for the computational analysis of these more realistic—and complex—environments.

A Proofs of Theorems

The appendix is organized as follows. We first derive a trio of lemmas that establish continuity properties and necessary conditions for solutions to the optimization problem of the proposer. We then state Lemma 4, which ensures that every feasible proposal under deferential voting can be approximated by a sequence of feasible proposals that are strictly preferred by the members of some decisive coalition. We proceed to define the mapping ψ, described in Section 4, and with Lemma 5 we establish that this mapping is continuous and that its domain and range can be restricted to a compact set. We then prove existence of legislative equilibrium in Theorem 1 by an application of Glicksberg's theorem, and parts 1–4 of the theorem follow immediately from Lemmas 1–3. In Lemma 6, we show that all legislative equilibrium continuation values are fixed points ψ. The proof of Theorem 2, which reduces all mixed legislative equilibria to pure, relies mainly on Lemmas 1 and 4. Theorem 3, on upper hemicontinuity of the equilibrium correspondence, follows from Lemmas 5 and 6. Theorem 4 uses the continuity of optimal proposals, proved in Lemma 3, along with known results on ergodicity of Markov chains. Finally, Theorem 5 follows from Theorems 1 and 5, with the help of Lemma 5.

Let $C^r(\Re^d, \Re^n)$ be the r-times continuously differentiable functions from $\Re^d$ into $\Re^n$ with the topology of C^r-uniform convergence on compacta. To describe this topology, let $\hat{r}$ be a non-negative integer and $Y \subseteq \Re^d$, and define the norm $||\phi||_{\hat{r},Y}$ on $C^{\hat{r}}(\Re^d, \Re^n)$ as $\sup\{||\partial\phi(x)|| : x \in Y\}$, where $\partial\phi$ is the $\hat{r}$-th derivative of ϕ. Then a sequence $\{\phi^m\}$ of functions converges to ϕ in $C^r(\Re^d, \Re^n)$ if and only if for every $\hat{r} = 0, 1, \ldots, r$ and every compact set $Y \subseteq \Re^d$, we have $||\phi^m - \phi||_{\hat{r},Y} \to 0$. We say $\phi^m \to \phi$ in $C^\infty(\Re^d, \Re^n)$ if and only if it converges in $C^r(\Re^d, \Re^n)$ for all $r = 0, 1, \ldots$. Given $v = (v_1, \ldots, v_n) \in C^r(\Re^d, \Re^n)$, define the induced utility

$$U_i(y, \theta_i; v) = (1 - \delta_i)(u_i(y) + \theta_i \cdot y) + \delta_i v_i(y),$$

where future payoffs are assumed to be generated by v, and define the associated acceptance sets

$$A_i(q, \theta; v) = \{y \in X \cup \{q\} : U_i(y, \theta_i; v) \geq U_i(q, \theta_i; v)\}.$$

Let $C \subseteq N$ be any coalition and $\mathscr{C} \subseteq 2^N$ any nonempty collection of coalitions, and, following the conventions of Section 3, define

$$A_C(q, \theta; v) = \bigcap_{i \in C} A_i(q, \theta; v) \quad \text{and} \quad A_{\mathscr{C}}(q, \theta; v) = \bigcup_{C \in \mathscr{C}} \bigcap_{i \in C} A_i(q, \theta; v).$$

When $C = \emptyset$, we adopt the convention that $A_C(q, \theta; v) = X \cup \{q\}$. Lastly, let

$$\max_y U_i(y, \theta; v) \qquad \mathscr{P}_i(\mathscr{C}, q, \theta; v)$$
$$\text{s.t. } y \in A_{\mathscr{C}}(q, \theta; v)$$

be the optimal proposal problem of legislator i, given status quo q and preference shocks θ, if the collection of decisive coalitions were $\mathscr{C}$ and continuation values were v. When $\mathscr{C}$ consists of a single coalition, C, we use the obvious shorthand $\mathscr{P}_i(C, q, \theta; v)$, substituting C for $\mathscr{C}$ in the notation defined above. Henceforth, the vector of functions v will be assumed to range over $C^r(\Re^d, \Re^n)$, unless otherwise restricted.

Our first lemma establishes, among other things, that the legislators' optimal proposals are essentially unique.

Lemma 1

1. *For all $\mathscr{C}$, the correspondence $A_{\mathscr{C}} \colon \Re^d \times \Theta \times C^r(\Re^d, \Re^n) \rightrightarrows \Re^d$ has nonempty, compact values and closed graph in (q, θ, v),*
2. *Fix $v \in C^0(\Re^d, \Re^n)$. For all i and all $\mathscr{C}$, there is a measurable function $\pi_i^{\mathscr{C}}(\cdot; v) \colon \Re^d \times \Theta \to \Re^d$ such that for all q and all θ, $\pi_i^{\mathscr{C}}(q, \theta; v)$ solves $\mathscr{P}_i(\mathscr{C}, q, \theta; v)$,*
3. *Fix $v \in C^0(\Re^d, \Re^n)$. For all q, there is a measure zero set $\Theta_1(q; v) \subseteq \Theta$ such that for all $\theta \notin \Theta_1(q; v)$, all i, and all $\mathscr{C}$, $\pi_i^{\mathscr{C}}(q, \theta; v)$ is the unique solution to $\mathscr{P}_i(\mathscr{C}, q, \theta; v)$.*

Proof We have $A_{\mathscr{C}}(q, \theta; v) \neq \emptyset$ for all (q, θ, v), as the status quo q belongs to $A_i(q, \theta; v)$ for all $i \in N$. By Mas-Colell's (1985) Theorem K.1.2, the function U_i is jointly continuous in (y, θ_i, v), and it follows that the correspondence A_i has closed graph in (q, θ, v). Compactness of $A_{\mathscr{C}}(q, \theta; v)$ follows since it is a closed subset of $X \cup \{q\}$, a compact set. This completes the proof of part 1. To prove part 2, fix $v \in C^0(\Re^d, \Re^n)$, and consider any i and $\mathscr{C}$. Then $U_i(\cdot; v)$ is a Caratheodory function, and Aliprantis and Border's (1999) Theorem 17.18 yields a measurable selection $\pi_i^{\mathscr{C}}(\cdot; v) \colon \Re^d \times \Theta \to \Re^d$ from the correspondence of solutions to $\mathscr{P}_i(\mathscr{C}, q, \theta; v)$. To prove part 3, fix $v \in C^0(\Re^d, \Re^n)$, and consider any q, any i, and any $\mathscr{C}$. Given preference shocks θ_{-i}, let

$$A_{\mathscr{C}}^{-i}(q, \theta_{-i}; v) = \bigcup_{C \in \mathscr{C}} A_{C \setminus \{i\}}(q, \theta; v)$$

denote the set of policies acceptable to all members, except possibly i, of some coalition in the collection $\mathscr{C}$. Note that if y solves $\mathscr{P}_i(\mathscr{C}, q, \theta; v)$, then it also solves

$$\max_y U_i(y, \theta_i; v)$$
$$\text{s.t., } y \in A_{\mathscr{C}}^{-i}(q, \theta_{-i}; v).$$

Note that if $y \neq y'$, then $D_{\theta_i}[U_i(y, \theta_i; v) - U_i(y', \theta_i; v)] = y - y' \neq 0$. Thus, Mas-Colell's (1985) Theorem I.3.1 implies that there is a measure zero set $\Theta_1^{i,\mathscr{C},\theta_{-i}}(q; v) \subseteq \Re^d$ such that for all $\theta_i \notin \Theta_1^{i,\mathscr{C},\theta_{-i}}(q; v)$, the program $\mathscr{P}_i(\mathscr{C}, q, \theta; v)$ admits a unique solution. Then

$$\Theta_1^{i,\mathcal{C}}(q; v) \;=\; \bigcup_{\theta_{-i} \in \Re^{(n-1)d}} \left(\Theta_1^{i,\mathscr{C},\theta_{-i}}(q; v) \times \{\theta_{-i}\}\right)$$

is measure zero. Finally, since N is finite,

$$\Theta_1(q;v) \;=\; \bigcup_{i\in N}\bigcup_{\mathscr{C}\subseteq 2^N}\Theta_1^{i,\mathscr{C}}(q;v)$$

is measure zero, as desired. ∎

Before we state the next lemma, we develop necessary notation and recall some definitions. For the moment, fix continuation values v and status quo q. For any subsets $C \subseteq N$ and $L \subseteq K$, define the functions $U^C\colon \Re^d \times \Theta \to \Re^{|C|}$ by $U^C(y,\theta;q,v) = (U_j(y,\theta_j;v) - U_j(q,\theta_j;v))_{j\in C}$ and $h^L\colon \Re^d \to \Re^{|L|}$ by $h^L(y) = (h_\ell(y))_{\ell\in L}$. Define the mapping $F^{C,L}\colon (\Re^d \setminus \{q\}) \times \Theta \to \Re^{|C|+|L|}$ by

$$F^{C,L}(y,\theta;q,v) = \begin{bmatrix} U^C(y,\theta;q,v) \\ h^L(y) \end{bmatrix},$$

where here (and whenever relevant) we view vectors as column matrices, making $F^{C,L}(y,\theta;q,v)$ a $(|C|+|L|)\times 1$ matrix. Derivatives are expanded via rows, e.g., $D_yU^C(y,\theta;q,v)$ is a $|C|\times d$ matrix. Define the mapping $\mathscr{L}_i^{C,L}\colon \Re^d \times \Re^{|C|+|L|} \times \Theta \to \Re$ by

$$\mathscr{L}_i^{C,L}(y,\lambda,\theta;q,v) = U_i(y,\theta_i;v) + \sum_{j\in C}\lambda_j(U_j(y,\theta_j;v) - U_j(q,\theta_j;v)) + \sum_{\ell\in L}\lambda_\ell h_\ell(y).$$

Let $G_i^{C,L}\colon (\Re^d \setminus \{q\}) \times \Re^{|C|+|L|} \times \Theta \to \Re^{d+|C|+|L|}$ be given by

$$G_i^{C,L}(y,\lambda,\theta;q,v) = \begin{bmatrix} D_y\mathscr{L}_i^{C,L}(y,\lambda,\theta;q,v)^T \\ -F^{C,L}(y,\theta;q,v) \end{bmatrix}.$$

Lastly, for any $m \in C \cup L$, define the mapping $G_i^{C,L,m}\colon (\Re^d \setminus \{q\}) \times \Re^{|C|+|L|} \times \Theta \to \Re^{d+|C|+|L|+1}$ by

$$G_i^{C,L,m}(y,\lambda,\theta;q,v) = \begin{bmatrix} G_i^{C,L}(y,\lambda,\theta;q,v) \\ \lambda_m \end{bmatrix}.$$

With regard to the program $\mathscr{P}_i(C,q,\theta;v)$, consider $y \in A_C(q,\theta;v)$ and let $\overline{C} \subseteq C$ and $\overline{K} \subseteq K$, with $K^{eq} \subseteq \overline{K}$, represent the voting and feasibility constraints, respectively, that hold with equality at y. We suppress the dependence of these sets on the pair (q,θ). Taking the coalition C as fixed, we say that y...

- satisfies the *linear independence constraint qualification (LICQ)* at (q,θ) if $D_yF^{\overline{C},\overline{K}}(y,\theta;q,v)$ has full row rank,
- is a *stationary solution* at (q,θ) if there exists $\lambda \in \Re^{|C|+|K|}$ such that $D_y\mathscr{L}_i^{C,K}(y,\lambda,\theta;q,v) = 0$, $\lambda_m \geq 0$ for all $m \in \overline{C} \cup (\overline{K} \cap K^{in})$, and $\lambda_m = 0$ for all $m \in (C \cup K) \setminus (\overline{C} \cup \overline{K})$,
- satisfies *strict complementary slackness* at (q,θ) if it is a stationary solution with multipliers λ such that $\lambda_m > 0$ for all $m \in \overline{C} \cup (\overline{K} \cap K^{in})$,
- satisfies the *second order sufficient conditions* with strict complementary slackness at (q,θ) if it is a stationary point and for all $z \in \Re^d \setminus \{0\}$ such that $D_yF^{\overline{C},\overline{K}}(y,\theta;q,v)z = 0$, we have

$$z^TD_{yy}\mathscr{L}_i^{C,K}(y,\lambda,\theta;q,v)z \;<\; 0.$$

In addition, we use the concept of strong stability from Kojima (1980). For this, we now explicitly parameterize the optimal proposal problem of legislator i by the functions U_j, $j \in N$, and h_ℓ, $\ell \in K$, as in $\mathscr{P}_i(C, q, \theta, (U_j)_{j\in N}, (h_\ell)_{\ell\in K})$. Let y be a stationary solution of $\mathscr{P}_i(C, q, \theta, (U_j)_{j\in N}, (h_\ell)_{\ell\in K})$, and let $B_\epsilon(y)$ be the open ball around y of radius $\epsilon > 0$. We say that y is *strongly stable* at (q, θ) with respect to a class $\mathscr{F} \subseteq C^r(\Re^d, \Re^{n+k})$ of perturbations if for some $\epsilon^* > 0$ and all $\epsilon \in (0, \epsilon^*]$, there exists $\eta > 0$ such that for all $\phi \in \mathscr{F}$ with $||\phi||_{2,B_{\epsilon^*}(y)} < \eta$, the optimization problem $\mathscr{P}_i(C, q, \theta, (U_j + \phi_j)_{j\in N}, (h_\ell + \phi_\ell)_{\ell\in K})$ has a stationary solution $y' \in B_\epsilon(y)$ that is unique in $B_{\epsilon^*}(y)$.

The main contribution of the next lemma is the continuity properties of the optimal proposal mapping, π_i^C, for an arbitrarily fixed coalition C. In fact, the lemma establishes that for all q and almost all θ, $\pi_i^C(q, \theta; v)$ is strongly stable, and $\pi_i^C(\cdot; v)$ is continuously differentiable in an open set.

Lemma 2 *Fix v. For all q, there exists a measure zero set $\Theta_2(q; v) \subseteq \Theta$ such that for all $\theta \notin \Theta_2(q; v)$, all i, and all C, the following hold.*

1. *Every $y \in A_C(q, \theta; v) \setminus \{q\}$ satisfies LICQ at (q, θ),*
2. *If $\pi_i^C(q, \theta; v) \neq q$, then it is a stationary solution satisfying the second order sufficient condition with strict complementary slackness at (q, θ),*
3. *If $\pi_i^C(q, \theta; v) \neq q$, then it is strongly stable at (q, θ) with respect to $\mathscr{F} = C^2(\Re^d, \Re^{n+k})$,*
4. *If $\pi_i^C(q, \theta; v) \neq q$, then there exists an open set $Z \subseteq \Re^d \times \Theta$ containing (q, θ) such that $\pi_i^C(\cdot; v)$ is continuously differentiable on Z and for all $(q', \theta') \in Z$, $\pi_i^C(q', \theta'; v)$ is the unique solution to $\mathscr{P}_i(C, q', \theta'; v)$.*

Proof Fix v, and consider any q. The proof is based on three applications of the transversality theorem (see Mas-Colell's (1985) Theorem I.2.2) to the mappings $F^{\overline{C},\overline{K}}$, $G_i^{\overline{C},\overline{K},m}$, and $G_i^{\overline{C},\overline{K}}$. First, consider any $\overline{C} \subseteq N$ and $\overline{K} \subseteq K$ such that $\overline{C} \cup \overline{K} \neq \emptyset$. The derivative of the mapping $F^{\overline{C},\overline{K}}(\cdot; q, v)$ at $(y, \theta) \in \Re^d \setminus \{q\} \times \Theta$ is the $(|\overline{C}| + |\overline{K}|) \times (d + |\overline{C}|d + (n - |\overline{C}|)d)$ matrix

$$DF^{\overline{C},\overline{K}}(y, \theta; q, v) \quad = \quad \begin{bmatrix} D_y U^{\overline{C}}(y, \theta; q, v) & (1 - \delta_i)(y - q)^T \otimes I_{|\overline{C}|} & 0 \\ Dh^{\overline{K}}(y) & 0 & 0 \end{bmatrix},$$

where $\otimes$ denotes Kronecker product. Since $y \neq q$ and $\delta_i < 1$, the rows of $(1 - \delta_i)(y - q)^T \otimes I_{|\overline{C}|}$ are linearly independent. For all (y, θ) such that $F^{\overline{C},\overline{K}}(y, \theta; q, v) = 0$, $\overline{K}$ is contained in the binding feasibility constraints at y, and therefore the rows of $Dh^{\overline{K}}(y)$ are linearly independent by assumption. Thus, $DF^{\overline{C},\overline{K}}(y, \theta; q, v)$ has full row rank. We conclude that $F^{\overline{C},\overline{K}}$ is transversal to $\{0\}$. For each θ, define $F_\theta^{\overline{C},\overline{K}} : \Re^d \setminus \{q\} \to \Re^{|\overline{C}|+|\overline{K}|}$ by $F_\theta^{\overline{C},\overline{K}}(y; q, v) = F^{\overline{C},\overline{K}}(y, \theta; q, v)$. Note that $F^{\overline{C},\overline{K}}$ is r-times continuously differentiable, where $r \geq d > \max\{0, d - (|\overline{C}| + |\overline{K}|)\}$. Thus, it follows by the transversality theorem that for almost all θ, $F_\theta^{\overline{C},\overline{K}}$ is transversal to $\{0\}$. Let $\hat{\Theta}_2^{\overline{C},\overline{K}}(q; v)$ be the measure zero set of θ's where this does not hold, and let $\hat{\Theta}_2(q; v)$ be the finite union of these sets over all $\overline{C}$ and $\overline{K}$ with $\overline{C} \cup \overline{K} \neq \emptyset$, which also has measure zero.

We proceed similarly for the mappings $G_i^{\overline{C},\overline{K},m}$ and $G_i^{\overline{C},\overline{K}}$. Consider any $\overline{C} \subseteq N \setminus \{i\}$, any $\overline{K}$, and any $m \in \overline{C} \cup \overline{K}$. The derivative $DG_i^{\overline{C},\overline{K},m}(y, \lambda, \theta; q, v)$ is the $(d + |\overline{C}| + |\overline{K}| + 1) \times (d + (|\overline{C}| +$

$|\overline{K}|) + d + (n-1)d)$ matrix

$$\begin{bmatrix} D_{yy}\mathscr{L}_i^{\overline{C},\overline{K}}(y,\lambda,\theta;q,v) & D_yF^{\overline{C},\overline{K}}(y,\theta;q,v)^T & (1-\delta_i)I_d & ((1-\delta_i)\lambda_j I_d)_{j\in\overline{C}} \\ -D_yU^{\overline{C}}(y,\theta;q,v) & 0 & 0 & (1-\delta_i)(q-y)^T\otimes I_{|\overline{C}|} \\ -Dh^{\overline{K}}(y) & 0 & 0 & 0 \\ 0 & 0\cdots 1\cdots 0 & 0 & 0 \end{bmatrix}.$$

(We omit zero-columns corresponding to derivatives with respect to θ_j, $j \notin \overline{C}\cup\{i\}$.) It is evident that for all (y,λ,θ) such that $G_i^{\overline{C},\overline{K},m}(y,\lambda,\theta;q,v) = 0$, this derivative has full row rank. Indeed, since $\delta_i < 1$, the rows of $(1-\delta_i)I_d$ are linearly independent; since $y \neq q$, the rows of $(1-\delta_i)(y-q)^T\otimes I_{|C|}$ are linearly independent; since $h^{\overline{K}}(y) = 0$, the rows of $Dh^{\overline{K}}(y)$ are linearly independent by assumption; and of course $\{(0\cdots 1\cdots 0)\}$ is linearly independent. Then, since $G_i^{\overline{C},\overline{K},m}$ is r-times continuously differentiable and $r \geq d > \max\{0, d+|\overline{C}|+|\overline{K}|-(d+|\overline{C}|+|\overline{K}|+1)\}$, the transversality theorem ensures that for almost all $\theta\in\Theta$, the mapping $G_{i,\theta}^{\overline{C},\overline{K},m}\colon(\Re^d\setminus\{q\})\times\Re^{|\overline{C}|+|\overline{K}|}\to\Re^d\times\Re^{|\overline{C}|+|\overline{K}|}\times\Re$ defined by $G_{i,\theta}^{\overline{C},\overline{K},m}(y,\lambda;q,v) = G_i^{\overline{C},\overline{K},m}(y,\lambda,\theta;q,v)$ is transversal to $\{0\}$. Since the dimension of the domain of $G_{i,\theta}^{\overline{C},\overline{K},m}$ is smaller than that of the range, the preimage theorem (see Mas-Colell's (1985) Theorem H.2.2) implies that $(G_{i,\theta}^{\overline{C},\overline{K},m})^{-1}(\{0\})$ is empty for almost all $\theta\in\Theta$, i.e., outside a set $\tilde{\Theta}_i^{\overline{C},\overline{K},m}(q;v)$ of measure zero, $G_i^{\overline{C},\overline{K},m}(y,\lambda,\theta;q,v)\neq 0$ for all (y,λ). Furthermore, defining $G_{i,\theta}^{\overline{C},\overline{K}}\colon(\Re^d\setminus\{q\})\times\Re^{|\overline{C}|+|\overline{K}|}\to\Re^d\times\Re^{|\overline{C}|+|\overline{K}|}$ by $G_{i,\theta}^{\overline{C},\overline{K}}(y,\lambda;q,v) = G_i^{\overline{C},\overline{K}}(y,\lambda,\theta;q,v)$, a third application of the transversality theorem to the mapping $G_i^{\overline{C},\overline{K}}$ ensures that outside a set $\tilde{\Theta}_i^{\overline{C},\overline{K}}(q;v)$ of measure zero, $G_{i,\theta}^{\overline{C},\overline{K}}(y,\lambda;q,v)$ is also transversal to $\{0\}$. Thus, we conclude that outside a measure zero set $\tilde{\Theta}_i^{\overline{C},\overline{K},m}(q;v)\cup\tilde{\Theta}_i^{\overline{C},\overline{K}}(q;v)$, every solution (y,λ) to $G_i^{\overline{C},\overline{K}}(y,\lambda,\theta;q,v) = 0$ is such that $D_{(y,\lambda)}G_i^{\overline{C},\overline{K}}(y,\lambda,\theta;q,v)$ is non-singular and $\lambda_m\neq 0$. Let $\tilde{\Theta}_2(q;v)$ be the finite union of these sets over all i, $\overline{C}\subseteq N\setminus\{i\}$, $\overline{K}$, and $m\in\overline{C}\cup\overline{K}$, which also has measure zero.

Now define $\Theta_2(q;v) = \Theta_1(q;v)\cup\hat{\Theta}_2(q;v)\cup\tilde{\Theta}_2(q;v)$, take any $\theta\notin\Theta_2(q;v)$, and consider any $i\in N$ and any $C\subseteq N$. To prove part 1, consider any $y\in A_C(q,\theta;v)\setminus\{q\}$, and let $\overline{C}\subseteq C$ and $\overline{K}\subseteq K$ represent the constraints that hold with equality at y. Trivially, y satisfies LICQ at (q,θ) if $\overline{C}\cup\overline{K} = \emptyset$. If $\overline{C}\cup\overline{K}\neq\emptyset$, then since $y\neq q$, we have $y\in(F_\theta^{\overline{C},\overline{K}})^{-1}(\{0\})$, and since $\theta\notin\hat{\Theta}_2(q;v)$, it follows that $DF_\theta^{\overline{C},\overline{K}}(y;q,v)$ has full rank. This completes the proof of part 1. To simplify notation in the remainder of the proof, let $y^* = \pi_i^C(q,\theta;v)\neq q$. By part 1, y^* satisfies LICQ, and hence, by Fiacco and McCormick's (1990) Corollaries 1 and 3, it is a stationary solution at (q,θ). Thus, let $\lambda^*\in\Re^{|C|+|K|}$ be the corresponding Lagrange multipliers (unique by LICQ), let $\overline{C}$ and $\overline{K}$ be the constraints that hold with equality at y^*, and let $\overline{\lambda}$ represent the coordinates of λ^* that correspond to $\overline{C}\cup\overline{K}$. Since $\theta\notin\Theta_1(q;v)$ and $y^*\neq q$, we must have $i\notin\overline{C}$, for otherwise $q\in A_C(q,\theta;v)$ also solves $\mathscr{P}_i(C,q,\theta;v)$, contradicting part 3 of Lemma 1. By definition of a stationary solution, we have $D_y\mathscr{L}_i^{C,K}(y^*,\lambda^*,\theta;q,v) = 0$ and $\lambda_m^* = 0$ for all $m\in(C\cup K)\setminus(\overline{C}\cup\overline{K})$. Thus, $D_y\mathscr{L}_i^{\overline{C},\overline{K}}(y^*,\overline{\lambda},\theta;q,v) = D_y\mathscr{L}_i^{C,K}(y^*,\lambda^*,\theta;q,v) = 0$, which implies $G_{i,\theta}^{\overline{C},\overline{K}}(y^*,\overline{\lambda};q,v) = 0$. But since $\theta\notin\tilde{\Theta}_2(q;v)$, we have $G_{i,\theta}^{\overline{C},\overline{K},m}(y^*,\overline{\lambda};q,v)\neq 0$ for all $m\in\overline{C}\cup\overline{K}$, and we conclude that $\lambda_m^* > 0$ for all $m\in\overline{C}\cup\overline{K}$. This establishes strict complementary slackness.

We have now deduced that y^* is a stationary point satisfying LICQ and strict complementary slackness at (q,θ). Note that $D_{(y,\lambda)}G_i^{\overline{C},\overline{K}}(y^*,\overline{\lambda},\theta;q,v)$ is given by the matrix

$$\begin{bmatrix} D_{(y,y)}\mathscr{L}_i^{C,K}(y^*,\lambda^*,\theta;q,v) & D_yF^{\overline{C},\overline{K}}(y^*,\theta;q,v)^T \\ -D_yF^{\overline{C},\overline{K}}(y^*,\theta;q,v) & 0 \end{bmatrix}, \tag{5}$$

which is non-singular since $\theta \notin \tilde{\Theta}_2(q;v)$. Thus, by Corollary 4.3 and equation (3-7) in Kojima (1980), y^* is strongly stable, establishing part 3.[24] By continuity, there exist open subsets $Z_1 \subseteq \Re^d \times \Theta$ and $W_1 \subseteq \Re^d$ with $(q,\theta,y^*) \in Z_1 \times W_1$ such that for all $(q',\theta',y') \in Z_1 \times W_1$, y' satisfies LICQ at (q',θ'). Since y^* is strongly stable and satisfies LICQ at (q,θ), Kojima's (1980) Corollary 6.6 implies that y^* satisfies the second order sufficient condition with strict complementary slackness.[25] This completes the proof of part 2.

The remainder of the proof addresses part 4. Since y^* is strongly stable and the data of the optimization problem are at least twice differentiable in (q,θ), Kojima's (1980) Theorem 8.2 yields open sets $Z_2 \subseteq Z_1$ and $W_2 \subseteq W_1$ with $(q,\theta,y^*) \in Z_2 \times W_2$ and a continuous function $\hat{\pi}\colon Z_2 \to W_2$ such that for all $(q',\theta') \in Z_2$, $\hat{\pi}(q',\theta')$ is the unique stationary solution of $\mathscr{P}_i(C,q',\theta';v)$ in W_2. With the second order sufficient condition, Fiacco's (1976) Theorem 2.1 ensures that the function $\hat{\pi}$ is continuously differentiable. Now, since y^* is the unique maximizer of $\mathscr{P}_i(C,q,\theta;v)$, we claim that there exist open sets $Z \subseteq Z_2$ and $W \subseteq W_2$ such that $(q,\theta,y^*) \in Z \times W$, and for all $(q',\theta') \in Z$, every solution of $\mathscr{P}_i(C,q',\theta';v)$ is contained in W. In fact, we may set $W = W_2$. Indeed, if the claim did not hold, then we could find a sequence $\{(q^m,\theta^m)\}$ converging to (q,θ) and a corresponding sequence of solutions $\{y^m\}$ to $\mathscr{P}_i(C,q^m,\theta^m;v)$ such that $y_m \notin W$ for all m. Since $q^m \to q$, it follows that $\overline{X} = X \cup \{q^m : m = 1,2,\ldots\} \cup \{q\}$ is compact. Since $y^m \in \overline{X} \setminus W$ for all m, we may go to a subsequence, still indexed by m, such that $y^m \to y$ for some $y \in \overline{X} \setminus W$. By part 1 of Lemma 1, A_C has closed graph, which implies that $y \in A_C(q,\theta;v)$, and therefore, by $\theta \notin \Theta_1(q;v)$ and part 3 of Lemma 1, we have $U_i(y^*,\theta_i;v) > U_i(y,\theta_i;v)$. But continuity of $\hat{\pi}$ implies that $\hat{\pi}(q^m,\theta^m) \to y^*$, and therefore $U_i(\hat{\pi}(q^m,\theta^m),\theta_i^m;v) > U_i(y^m,\theta_i^m;v)$ for high enough m. Since $\hat{\pi}(q^m,\theta^m) \in A_C(q^m,\theta^m;v)$ for all m, this contradicts optimality of y^m. For all $(q',\theta') \in Z$, every solution y' to $\mathscr{P}_i(C,q',\theta';v)$ is such that $y' \in W$. By Corollaries 1 and 3 in Fiacco and McCormick (1990), since y' satisfies LICQ at (q',θ'), it follows that y' is a stationary solution of $\mathscr{P}_i(C,q',\theta';v)$. But $\hat{\pi}(q',\theta')$ is the only stationary solution of $\mathscr{P}_i(C,q',\theta';v)$ in $W_2 \supseteq W$, so we conclude that for all $(q',\theta') \in Z$, we have $\pi_i^C(q',\theta';v) = \hat{\pi}(q',\theta') \in W$, and therefore $\pi_i^C(\cdot;v)$ is continuously differentiable in an open set around (q,θ), as desired. ∎

The next lemma shows that for all q and almost all θ, there exists a coalition C^* such that in an open set around (q,θ), the solution of $\mathscr{P}_i(\mathscr{C},q,\theta;v)$ reduces to the optimal proposal subject to approval by the members of C^*. Thus, on an open set of (q,θ) pairs with full measure, the maximizer $\pi_i^{\mathscr{C}}(q,\theta;v)$ inherits the differentiability properties of Lemma 3. The lemma also shows that if any legislator is indifferent between the optimal proposal and the status quo, then all such

[24] Kojima's Condition 1.1 is fulfilled by part 1 of the lemma. By his Corollary 4.3, strong stability follows if his mapping F is locally nonsingular at z^*, where z^* is y^* coupled with generalized multipliers $\tilde{\lambda}$. Kojima's Theorem 3.3 gives complicated conditions for local nonsingularity that involve sets J of constraints satisfying $J_+(\tilde{\lambda}) \subseteq J \subseteq J_n(\tilde{\lambda})$. In our context, strict complementary slackness pins down J to the set of binding inequality constraints. We use Kojima's sufficient condition that, with the latter restriction, the determinant of $DF(z^*;\tau(J))$ is nonzero. In equation (3-7), he gives a simplified form of that determinant, where A is the matrix of gradients of binding equality and inequality constraints. His condition is satisfied if our matrix in (5) is nonsingular.

[25] Kojima's corollary shows that since y^* is a solution to $\mathscr{P}_i(C,q,\theta;v)$, then y^* together with generalized multipliers $\tilde{\lambda}$ satisfies his Condition 6.2. This condition states that $D_{yy}\mathscr{L}_i^{C,K}(y^*,\lambda^*,\theta;q,v)$ is negative definite on a subspace given by constraints $L \cup J_+(\tilde{\lambda})$. In our context, strict complementary slackness implies that these constraints are exactly $\overline{C} \cup \overline{K}$, which yields the second order sufficient condition.

legislators are necessary in order for the proposal to be approved by a coalition in $\mathscr{C}$: if one, say j, is removed, then the resulting coalition, $C^* \setminus \{j\}$, no longer belongs to $\mathscr{C}$.

Lemma 3 *Fix v. For all q, there is a measure zero set $\Theta_3(q;v) \subseteq \Theta$ such that for all $\theta \notin \Theta_3(q;v)$, all i, and all $\mathscr{C}$, if $\pi_i^{\mathscr{C}}(q,\theta;v) \neq q$ and we define*

$$C^* \;=\; \{j \in N : U_j(\pi_i^{\mathscr{C}}(q,\theta;v),\theta_j;v) \geq U_j(q,\theta_j;v)\},$$

then there is an open set Z containing (q,θ) such that the following hold.

1. *For all nonempty C satisfying $U_j(\pi_i^{\mathscr{C}}(q,\theta;v),\theta_j;v) = U_j(q,\theta_j;v)$ for all $j \in C$, we have $C^* \setminus C \notin \mathscr{C}$.*
2. *For all $(q',\theta') \in Z$, the unique solution to $\mathscr{P}_i(\mathscr{C},q,\theta;v)$ is $\pi_i^{\mathscr{C}}(q',\theta';v) = \pi_i^{C^*}(q',\theta';v)$,*
3. *The optimal proposal mapping $\pi_i^{\mathscr{C}}(q',\theta';v)$ is continuously differentiable on Z.*

Proof Fix v, and consider any q. We claim that there is a measure zero set $\tilde{\Theta}_3(q;v)$ such that for all $\theta \notin \tilde{\Theta}_3(q;v)$, all i, all C, and all $j \notin C \cup \{i\}$, if $\pi_i^C(q,\theta;v) \neq q$, then $U_j(\pi_i^C(q,\theta;v),\theta_j;v) \neq U_j(q,\theta_j;v)$. Indeed, fix q and i arbitrarily, and consider any coalition C. Note that for all $j \notin C \cup \{i\}$, $\pi_i^C(q,\theta_j,\theta_{-j};v)$ is independent of θ_j. Thus, if θ_{-j} is such that $\pi_i^C(q,\theta;v) \neq q$, then the preference shocks θ'_j that solve the equality $U_j(\pi_i^C(q,(\theta_{-j},\theta'_j);v),\theta'_j;v) = U_j(q,\theta'_j;v)$ form a lower dimensional hyperplane in $\Re^d$. We infer that there is a measure zero set $\tilde{\Theta}_3^{i,C,j,\theta_{-j}}(q;v) \subseteq \Re^d$ such that for all $\theta_j \notin \tilde{\Theta}_3^{i,C,j,\theta_{-j}}(q;v)$, we have $U_j(\pi_i^C(q,(\theta_{-j},\theta_j);v),\theta_j;v) \neq U_j(q,\theta_j;v)$. Then

$$\tilde{\Theta}_3^{i,C,j}(q;v) \;=\; \bigcup \left\{\theta \in \Theta : \theta_j \in \tilde{\Theta}_3^{i,C,j,\theta_{-j}}(q;v), \pi_i^C(q,\theta;v) \neq q\right\}$$

is measure zero. Since N is finite,

$$\tilde{\Theta}_3(q;v) \;=\; \bigcup \left\{\tilde{\Theta}_3^{i,C,j}(q;v) : i \in N, C \subseteq N, j \notin C \cup \{i\}\right\}$$

is also measure zero, as desired.

We now define $\Theta_3(q;v) = \Theta_1(q;v) \cup \Theta_2(q;v) \cup \tilde{\Theta}_3(q;v)$. Consider any $\theta \notin \Theta_3(q;v)$, any i, and any $\mathscr{C}$, and suppose that $\pi_i^{\mathscr{C}}(q,\theta;v) \neq q$. Define C^* as in the statement of the lemma. To prove part 1, consider any nonempty C satisfying $U_j(\pi_i^{\mathscr{C}}(q,\theta;v),\theta_j;v) = U_j(q,\theta_j;v)$ for all $j \in C$. Since $\theta \notin \Theta_1(q;v)$, part 3 of Lemma 1 implies that $U_i(\pi_i^{\mathscr{C}}(q,\theta;v),\theta_i;v) > U_i(q,\theta_i;v)$, so that $i \notin C$. Suppose, to obtain a contradiction, that $C' = C^* \setminus C \in \mathscr{C}$, and take any $j \in C$. Note that $\pi_i^{\mathscr{C}}(q,\theta;v)$ solves $\mathscr{P}_i(C^*,q,\theta;v)$, and since $\mathscr{P}_i(C',q,\theta;v)$ removes at least one constraint, we have $U_i(\pi_i^{C'}(q,\theta;v),\theta_i;v) \geq U_i(\pi_i^{\mathscr{C}}(q,\theta;v),\theta_i;v)$. Since $C' \in \mathscr{C}$, we have $\pi_i^{C'}(q,\theta;v) \in A_{\mathscr{C}}(q,\theta;v)$. Then, since $\theta \notin \Theta_1(q;v)$, part 3 of Lemma 1 implies $\pi^{C'}(q,\theta;v) = \pi_i^{\mathscr{C}}(q,\theta;v) \neq q$. But then $j \notin C' \cup \{i\}$ and $U_j(\pi_i^{C'}(q,\theta;v),\theta_j;v) = U_j(\pi_i^{\mathscr{C}}(q,\theta;v),\theta_j;v) = U_j(q,\theta_j;v)$ contradicts $\theta \notin \tilde{\Theta}_3(q;v)$. We conclude that $C^* \setminus C \notin \mathscr{C}$.

To prove part 2, note that $\pi_i^{\mathscr{C}}(q,\theta;v) = \pi_i^{C^*}(q,\theta;v)$ and $C^* \in \mathscr{C}$. Suppose there does not exist an open set Z containing (q,θ) such that for all $(q',\theta') \in Z$, the unique solution to $\mathscr{P}_i(\mathscr{C},q,\theta;v)$ is $\pi_i^{\mathscr{C}}(q',\theta';v) = \pi_i^{C^*}(q',\theta';v)$. Then there are sequences $\{(q^m,\theta^m)\}$ and $\{y^m\}$ such that $(q^m,\theta^m) \to (q,\theta)$ and for all m, $\pi_i^{C^*}(q^m,\theta^m;v)$ is not the unique solution to $\mathscr{P}_i(\mathscr{C},q^m,\theta^m;v)$. By part 2 of Lemma 1, this program has at least one solution, so for all m, there exists $y^m \in A_{\mathscr{C}}(q^m,\theta^m;v) \setminus \{\pi_i^{C^*}(q^m,\theta^m;v)\}$ such that $U_i(y^m,\theta_i^m;v) \geq U_i(\pi_i^{C^*}(q^m,\theta^m;v),\theta_i^m;v)$. Part 4 of Lemma 2 yields an open set Z' around (q, θ) such that for all C, $\pi_i^C(\cdot; v)$ is continuous (in fact differentiable) on Z', and for all $(q', \theta') \in Z'$, $\pi_i^C(q', \theta'; v)$ is the unique solution to program $\mathscr{P}_i(C, q', \theta'; v)$. Thus, for high enough m, we have $y^m \in \{\pi_i^C(q^m, \theta^m; v) : C \in \mathscr{C}\}$, i.e., there exists $C^m \in \mathscr{C}$ such that $y^m = \pi_i^{C^m}(q^m, \theta^m; v)$. Since N is finite, we may assume a subsequence, still indexed by m, and a coalition $\hat{C} \in \mathscr{C}$ satisfying $C^m = \hat{C}$ for all m. Since $q^m \to q$, it follows that $\overline{X} = X \cup \{q^m : m = 1, 2, \ldots\} \cup \{q\}$ is compact. Since $y^m \in \overline{X}$ for all m, we may go to a further subsequence, still indexed by m, such that $y^m \to y \in \overline{X}$. In fact, since $y^m \in A_{\hat{C}}(q^m, \theta^m; v)$ for all m, part 1 of Lemma 1 implies that $y \in A_{\hat{C}}(q, \theta; v) \subseteq A_{\mathscr{C}}(q, \theta; v)$. Note that, as a consequence, $y = \pi_i^{C^*}(q, \theta; v)$, for otherwise, since $\theta \notin \Theta_1(q; v)$, part 3 of Lemma 1 implies that $U_i(\pi_i^{C^*}(q, \theta; v), \theta_i; v) > U_i(y, \theta_i; v)$. Since U_i is continuous and $\pi_i^{C^*}(\cdot; v)$ is continuous at (q, θ), we then have $U_i(\pi_i^{C^*}(q^m, \theta^m; v), \theta_i^m; v) > U_i(y^m, \theta_i^m; v)$ for high enough m. But $C^* \in \mathscr{C}$ implies $\pi_i^{C^*}(q^m, \theta^m; v) \in A_{\mathscr{C}}(q^m, \theta^m; v)$, contradicting optimality of y^m. Therefore, $y^m \to y = \pi_i^{C^*}(q, \theta; v)$. Note that for all $j \in \hat{C}$ and all m, we have $U_j(y^m, \theta_j^m; v) \geq U_j(q, \theta_j^m; v)$. By continuity, we then have $U_j(\pi_i^{C^*}(q, \theta; v), \theta_j; v) \geq U_j(q, \theta_j; v)$ for all $j \in \hat{C}$, which implies $\hat{C} \subseteq C^*$. This immediately implies $U_i(y^m, \theta_i^m; v) \geq U_i(\pi_i^{C^*}(q^m, \theta_i^m); \theta_i^m; v)$. Recall that $\pi_i^{\mathscr{C}}(q, \theta; v) = \pi_i^{C^*}(q, \theta; v)$. Thus, for all j with $U_j(\pi_i^{C^*}(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)$, we must have $j \in \hat{C}$, for otherwise part 1 of the lemma implies $\hat{C} \notin \mathscr{C}$, a contradiction. Now consider j with $U_j(\pi_i^{C^*}(q, \theta; v), \theta_j; v) > U_j(q, \theta_j; v)$. Since $y^m \to \pi_i^{C^*}(q, \theta; v)$ and $q^m \to q$, continuity of U_j implies that for high enough m, we have $U_j(y^m, \theta_j^m; v) > U_j(q^m, \theta_j^m; v)$. Thus, $y^m \in A_{C^*}(q^m, \theta^m; v)$. But since $\mathscr{P}_i(C^*, q', \theta'; v)$ has a unique solution in an open set around (q, θ), we then have $\pi_i^{C^*}(q^m, \theta^m) = y^m$ for high enough m. This final contradiction yields an open set Z with the desired properties and completes the proof of part 2. Part 3 then follows directly from part 4 of Lemma 2. ∎

The next lemma shows that, generically, any feasible policy x that is weakly preferred to the status quo by a decisive coalition of legislators can be approximated by feasible policies that are strictly preferred to the status quo by a decisive coalition.

Lemma 4 *Fix v. For all q, there is a measure zero set $\Theta_4(q; v) \subseteq \Theta$ such that for all $\theta \notin \Theta_4(q; v)$, all i, all $\mathscr{C}$, and all $y \in A_{\mathscr{C}}(q, \theta; v) \setminus \{q\}$, there exists a sequence $\{y^m\}$ in $A_{\mathscr{C}}(q, \theta; v)$ such that $y^m \to y$ and for all j and all m, $U_j(y^m, \theta_j; v) \neq U_j(q, \theta_j; v)$.*

Proof Fix v, consider any q, and define $\Theta_4(q; v) = \Theta_2(q; v)$. Consider any $\theta \notin \Theta_4(q; v)$, any i, any $\mathscr{C}$, and any $y \in A_{\mathscr{C}}(q, \theta; v) \setminus \{q\}$. Let $C^* = \{j \in N : U_j(y, \theta_j; v) \geq U_j(q, \theta_j; v)\}$. Since $y \in A_{\mathscr{C}}(q, \theta; v)$, we have $C^* \in \mathscr{C}$. Let $\overline{C}$, and $\overline{K}$ denote the voting and feasibility constraints, respectively, that bind at y in program $\mathscr{P}_i(C^*, q, \theta; v)$. Since $y \neq q$ and $\theta \notin \Theta_2(q; v)$, y satisfies LICQ at (q, θ). Define the mapping $F \colon \Re^{d+1} \to \Re^{|\overline{C}| + |\overline{K}|}$ by

$$F(x, \epsilon) \quad = \quad \left[\begin{array}{c} (U_j(x, \theta_j; v) - \epsilon - U_j(q, \theta_j; v))_{j \in \overline{C}} \\ (h_\ell(x))_{\ell \in \overline{K}} \end{array} \right],$$

and note that $F(y, 0) = 0$. By LICQ, $D_x F(y, 0)$ has full row rank, and the implicit function theorem (see Loomis and Sternberg (1968)) yields open sets $P \subseteq \Re$ around zero and $Y \subseteq \Re^d$ around y and a continuous mapping $\phi \colon P \to Y$ such that $\phi(0) = y$ and for all $\epsilon \in P$, $F(\phi(\epsilon), \epsilon) = 0$. Defining the sequence $\{y^m\}$ by $y^m = \phi(1/m)$, continuity of ϕ implies $y^m \to y$. For all $\ell \in K^{in} \setminus \overline{K}$, we have $h_\ell(y) > 0$, and continuity of h_ℓ then implies that for sufficiently high m, we have $h_\ell(y^m) > 0$. And $F(y^m, 1/m) = 0$ implies that for all $\ell \in \overline{K}$, we have $h_\ell(y^m) = 0$. Thus, $y^m \in X$ for sufficiently high m. For all $j \in C^* \setminus \overline{C}$, so that $U_j(y, \theta_j; v) > U_j(q, \theta_j; v)$, continuity of U_j implies that for sufficiently high m, we have $U_j(y^m, \theta_j; v) > U_j(q, \theta_j; v)$. And $F(y^m, 1/m) = 0$ implies that for all $j \in \overline{C}$, we have $U_j(y^m, \theta_j; v) - U_j(q, \theta_j; v) = 1/m > 0$. Therefore, $y^m \in A_{C^*}(q, \theta; v) \subseteq A_{\mathscr{C}}(q, \theta; v)$. Furthermore, for all $j \notin C^*$ such that $U_j(y, \theta_j; v) < U_j(q, \theta_j; v)$, continuity implies that for sufficiently high m, we have $U_j(y^m, \theta_j; v) < U_j(q, \theta_j; v)$. Therefore, we have established the existence of a subsequence $\{y^m\}$ in $A_{\mathscr{C}}(q, \theta; v)$, such that $y^m \to y$ and $U_j(y^m, \theta_j; v) \neq U_j(q, \theta_j; v)$ for all j, as required. ∎

As in Section 4, we now index models by $\gamma = ((p_i, u_i, \delta_i)_{i \in N}, X, f, g)$, we let Γ denote the metric space of parameterizations satisfying the assumptions of the legislative model, and we continue to assume that the parameterization is continuous in the sense of that section. We define the induced utility $U_i^\gamma(y, \theta_i; v)$ in model γ in the obvious way, and it is immediate that U_i is jointly continuous in (y, θ_i, v, γ). Given model $\gamma \in \Gamma$ and continuation value functions v, Lemma 1 allows us to define measurable mappings $\pi_i^\gamma(\cdot; v) \colon \Re^d \times \Theta \to \Re^d$ such that for all q and almost all θ, $\pi_i^\gamma(q, \theta; v)$ solves $\mathscr{P}_i^\gamma(\mathscr{D}, q, \theta; v)$, i.e., it solves the proposer's optimization problem at (q, θ) in model γ when the voting rule is given by $\mathscr{D}$ and continuation values are given by v. We use these optimal proposal mappings to define a best response continuation value mapping ψ as follows: define $\psi \colon C^0(\Re^d, \Re^n) \times \Gamma \to C^0(\Re^d, \Re^n)$ by

$$\psi(v, \gamma)(x) \quad = \quad \int_q \int_\theta \sum_j p_j^\gamma U_i^\gamma(\pi_j^\gamma(q, \theta; v), \theta_i; v) f^\gamma(\theta) g^\gamma(q|x) d\theta dq.$$

where $\psi(v, \gamma) \in C^0(\Re^d, \Re^n)$ follows from the fact that $\psi(v, \gamma)$ depends on x only through the density $g^\gamma(q|x)$, which is continuous.[26] When a model γ is fixed, we may write $\psi^\gamma(v)$ for the value $\psi(v, \gamma)$.

The next lemma establishes that the domain and range of ψ can be restricted to a compact space and that the mapping ψ is continuous on this space. Without loss of generality, assume $a, b \geq 1$. Define $\mathscr{V}$ to consist of functions $v \in C^r(\Re^d, \Re^n)$ such that (i) for all x and all i, $|v_i(x)| \leq c$, (ii) if $r < \infty$, then the derivatives of v of order $0, 1, \ldots, r$ are bounded in norm by $\sqrt{n}abc$, and the r-th derivative of v is Lipschitz continuous with modulus $\sqrt{n}abc$, and (iii) if $r = \infty$, then the derivatives of v of all orders $0, 1, 2, \ldots$ are bounded in norm by $\sqrt{n}abc$. Denote by $M(\Re^d, \Re^n)$ the set of Borel measurable mappings from $\Re^d$ to $\Re^n$.

Lemma 5

1. *The space $\mathscr{V}$ is nonempty, convex, and compact.*
2. *Fix $\gamma \in \Gamma$ and $\phi \in M(\Re^d, \Re^n)$ such that for all i, ϕ_i is bounded in absolute value by c over $\tilde{X}$. Define the mapping $\hat{\phi} \in M(\Re^d, \Re^n)$ by $\hat{\phi}(x) = \int_q \phi(q) g^\gamma(q|x) dq$ for all x. Then $\hat{\phi} \in \mathscr{V}$.*
3. *The mapping $\psi \colon \mathscr{V} \times \Gamma \to \mathscr{V}$ is continuous.*

Proof Clearly, $\mathscr{V}$ is nonempty and convex. We claim that it is compact. Given $\hat{r} < \infty$, let $\mathscr{V}^{\hat{r}}$ be the subspace of $C^{\hat{r}}(\Re^d, \Re^n)$ such that $v \in \mathscr{V}^{\hat{r}}$ if and only if all derivatives of v of order $0, 1, \ldots, \hat{r}$ are bounded in norm by $\sqrt{n}abc$. By Mas-Colell's (1985) Theorem K.2.2, $\mathscr{V}^{\hat{r}}$ is compact in the topology of $C^{\hat{r}}$-uniform convergence on compacta. Let $\mathscr{V}'$ be the subset of $C^0(\Re^d, \Re^n)$ such that $v \in \mathscr{V}'$ if and only if for all i, v_i is bounded in absolute value by c. This set is closed in the topology of C^0-uniform convergence on compacta and all finer topologies. Therefore, since $a, b \geq 1$, $r < \infty$ implies that $\mathscr{V} = \mathscr{V}' \cap \mathscr{V}^r$ is a closed subset of a compact space and, therefore, compact. And $r = \infty$ implies that $\mathscr{V} = \mathscr{V}' \cap \bigcap_{\hat{r}=1}^\infty \mathscr{V}^{\hat{r}}$ is compact in the topology of $C^{\hat{r}}$-uniform convergence on compacta for all $\hat{r} = 0, 1, \ldots$. Compactness of $\mathscr{V}$ then follows from Mas-Colell's Theorem K.2.2.1. This completes the proof of part 1.

For part 2, consider any $\gamma \in \Gamma$ and $\phi \in M(\Re^d, \Re^n)$ such that for all i, ϕ_i is bounded in absolute value by c. Define $\hat{\phi}$ as in the statement of the lemma. By Aliprantis and Burkinshaw's (1990)

[26] This follows from a stronger result established in part 2 of Lemma 5.

Theorem 20.4, each function $\hat{\phi}_i$ is partially differentiable. Let ∂^α denote a partial derivative operator with respect to the coordinates of x of any order $\hat{r} = 1, 2, \ldots, r$, with multi-index α. Aliprantis and Burkinshaw's result, with the expression in (1) implies that

$$\partial^\alpha \hat{\phi}(x) \;\; = \;\; \int_q \phi(q) \partial^\alpha g^\gamma(q|x) dq. \tag{6}$$

Since this depends on x only through $\partial^\alpha g^\gamma(q|x)$, which is continuous, it follows that $\partial^\alpha \hat{\phi}_i$ is continuous. Indeed, consider a sequence $\{x^m\}$ in $\Re^d$ converging to x. Then the integrand in $\partial^\alpha \hat{\phi}_i(x^m)$, as a function of q, converges pointwise to the integrand in $\partial^\alpha \hat{\phi}_i(x)$. Furthermore, we assume that the $\hat{r}$-th derivative of $g^\gamma(q|x)$ with respect to x is bounded in norm by b, which implies $|\partial^\alpha g^\gamma(q|x^m)| \leq b$ for all m. Since the support of $g^\gamma(\cdot|x)$ lies in $\tilde{X}$ for all x, it follows that $\partial^\alpha g^\gamma(\cdot|x)$ is identically zero outside $\tilde{X}$. Therefore, since ϕ is bounded in absolute value by c on $\tilde{X}$, we have $|\phi(q)\partial^\alpha g^\gamma(q|x)| \leq cbI_{\tilde{X}}(q)$ for all q, and the claimed continuity follows from Lebesgue's dominated convergence theorem. Therefore, $\hat{\phi}$ is r-times continuously differentiable. To prove that $\hat{\phi} \in \mathscr{V}$, we first note that, because it is the expectation of ϕ_i with respect to a density with support in $\tilde{X}$, $\hat{\phi}_i$ is bounded in absolute value by c over $\tilde{X}$ for all i. Thus, (i) is fulfilled. To verify (ii), suppose $r < \infty$, and let ∂ be a derivative operator of order $\hat{r} = 0, 1, \ldots, r$, where we view $\partial\hat{\phi}(x)$ as a $n \times d^{\hat{r}}$ matrix and $\partial g^\gamma(q|x)$ as a $1 \times d^{\hat{r}}$ row vector. Then, viewing $\hat{\phi}(x)$ and $\phi(q)$ as $n \times 1$ column vectors, we have from (6) that $\partial\hat{\phi}(x) = \int_q \phi(q)\partial g^\gamma(q|x)dq$, and consequently,

$$||\partial\hat{\phi}(x)|| \;\; \leq \;\; \int_q ||\phi(q)\partial g^\gamma(q|x)||dq \;\; \leq \;\; \int_q ||\phi(q)||\,||\partial g^\gamma(q|x)||dq,$$

where the first inequality follows from Jensen's inequality and the second follows from Aliprantis and Border's (1999) Lemma 6.6. Note that $||\phi(q)|| \leq \sqrt{n}c$. Again, $\partial g^\gamma(\cdot|x)$ is identically zero outside $\tilde{X}$, and we therefore have

$$||\partial\hat{\phi}(x)|| \;\; \leq \;\; \int_{\tilde{X}} \sqrt{n}c||\partial g^\gamma(q|x)||dq \;\; \leq \;\; \int_{\tilde{X}} \sqrt{n}bcdq \;\; = \;\; \sqrt{n}abc.$$

Now let ∂ denote the r-th order derivative with respect to x, and note that for all x and y,

$$\begin{aligned} ||\partial\hat{\phi}(x) - \partial\hat{\phi}(y)|| \;\; &= \;\; \left|\left|\int_q \phi(q)(\partial g^\gamma(q|x) - \partial g^\gamma(q|y))d\theta dq\right|\right| \\ &\leq \;\; \int_q ||\phi(q)||\,||\partial g^\gamma(q|x) - \partial g^\gamma(q|y)||dq \\ &\leq \;\; \int_{\tilde{X}} \sqrt{n}c||\partial g^\gamma(q|x) - \partial g^\gamma(q|y)||dq \\ &\leq \;\; \sqrt{n}abc||x-y||, \end{aligned}$$

where the last inequality follows from our assumption that the r-th derivative of $g^\gamma(q|x)$ with respect to x is Lipschitz continuous with modulus b. Therefore, $\partial\hat{\phi}$ is Lipschitz continuous with modulus $\sqrt{n}abc$, fulfilling (ii). Now suppose $r = \infty$, and consider any $\hat{r} \geq 1$. As argued above, we have $||\partial\hat{\phi}(x)|| \leq \sqrt{n}abc$, fulfilling (iii) and implying $\hat{\phi} \in \mathscr{V}$.

For part 3, first consider any $(v, \gamma) \in \mathscr{V} \times \Gamma$, and define the measurable mapping $w \colon \Re^d \to \Re^n$ by

$$w_i(q) \;\; = \;\; \int_\theta \sum_j p_j^\gamma U_i^\gamma(\pi_j^\gamma(q, \theta; v), \theta_i; v) f^\gamma(\theta) d\theta \tag{7}$$

for all i and all q. Recall that $|u_i^\gamma(x) + \theta_i \cdot x| f^\gamma(\theta) \le c$ for all i, all $\theta \in \Theta$, and all $x \in \tilde{X}$. Since $\pi_i^\gamma(q, \theta; v) \in X \cup \{q\}$, we then have for all i and all $q \in \tilde{X}$,

$$|w_i(q)| \;\; \le \;\; \int_\theta \sum_j p_j^\gamma \big[(1-\delta_i^\gamma)|u_i^\gamma(\pi_i^\gamma(q,\theta;v)) + \theta_i \cdot \pi_i^\gamma(q,\theta;v)| + \delta_i^\gamma |v_i(\pi_i^\gamma(q,\theta;v))|\big] d\theta \;\; \le \;\; c.$$

Noting that $\psi(v,\gamma)(x) = \int_q w(q) g^\gamma(q|x) dq$, it follows from part 2 of the lemma that $\psi(v,\gamma) \in \mathscr{V}$. We conclude that $\psi \colon \mathscr{V} \times \Gamma \to \mathscr{V}$, as desired.

To prove continuity, consider sequences $\{v^m\}$ in $\mathscr{V}$ and $\{\gamma^m\}$ in Γ with $v^m \to v^* \in C^r(\Re^d, \Re^n)$ and $\gamma^m \to \gamma^* \in \Gamma$. Write $\gamma^m = ((p_i^m, u_i^m, \delta_i^m)_{i \in N}, X^m, f^m, g^m)$ and $\gamma^* = ((p_i^*, u_i^*, \delta_i^*)_{i \in N}, X^*, f^*, g^*)$. We use superscript m for variables corresponding to model γ^m, and we use a superscript asterisk for variables corresponding to γ^*. We first note that there exists a compact set $\hat{X} \subseteq \Re^d$ such that for sufficiently high m, $X^m \subseteq \hat{X}$. Indeed, X^* is compact by assumption. Letting B be the closure of an open ball of finite, positive radius, it follows that $X^* + B$ is compact. Since $X^m \to X^*$ Hausdorff, we have $X^m \subseteq X^* + B$ for high enough m.

We claim that for all q, all i, and all $\theta \notin \Theta_1^*(q;v^*) \cup \Theta_2^*(q;v^*)$, $\pi_i^m(q,\theta;v^m) \to \pi_i^*(q,\theta;v^*)$. If not, then because $\pi_i^m(q,\theta;v^m)$ lies in the compact set $\hat{X}$ for high enough m, we may go to a subsequence, still indexed by m, such that $\pi_i^m(q,\theta;v^m) \to x \ne \pi_i^*(q,\theta;v^*)$. Since $\pi_i^m(q,\theta;v^m) \in A_{\mathscr{D}}^m(q,\theta;v^m)$ for all m, part 1 of Lemma 1 implies that $x \in A_{\mathscr{D}}^*(q,\theta;v^*)$. And since $\theta \notin \Theta_1^*(q;v^*)$, part 3 of Lemma 1 implies that $U_i^*(\pi_i^*(q,\theta;v^*), \theta_i; v^*) > U_i^*(x, \theta_i; v^*)$. We consider two cases. First, suppose $\pi_i^*(q,\theta;v^*) \ne q$, so by $\theta \notin \Theta_2^*(q;v^*)$, Lemma 4 implies that there exists $y \in A_{\mathscr{D}}^*(q,\theta;v^*)$ arbitrarily close to $\pi_i^*(q,\theta;v^*)$ such that $U_j^*(y,\theta_j;v^*) \ne U_j^*(q,\theta_j;v^*)$ for all j. Thus, there exists a decisive coalition $C \in \mathscr{D}$ such that $U_j^*(y,\theta_j;v^*) > U_j^*(q,\theta_j;v^*)$ for all $j \in C$. Furthermore, by $U_i^*(\pi_i^*(q,\theta;v^*),\theta_i;v^*) > U_i^*(x,\theta_i;v^*)$ and continuity of U_i^*, we may suppose $U_i^*(y,\theta_i;v^*) > U_i^*(x,\theta_i;v^*)$. Since $y \in X^*$, and since $X^m \to X^*$ Hausdorff, there exists a sequence $\{y^m\}$ in $\Re^d$ such that $y^m \in X^m$ for all m and $y^m \to y$. By joint continuity, we then have for all $j \in C$ and for high enough m, $U_j^m(y^m,\theta_j;v^m) > U_j^m(q,\theta_j;v^m)$, implying $y^m \in A_{\mathscr{D}}^m(q,\theta;v^m)$. But by joint continuity, we also have $U_i^m(y^m,\theta;v^m) > U_i^m(\pi_i^m(q,\theta;v^m),\theta_i;v^m)$ for high enough m, contradicting the fact that $\pi_i^m(q,\theta;v^m)$ solves $\mathscr{P}_i^m(\mathscr{D},q,\theta;v^m)$. For the second case, suppose $\pi_i^*(q,\theta;v^*) = q$. Then $\pi_i^*(q,\theta;v^*) = q \in A_{\mathscr{D}}^m(q,\theta;v^m)$ for all m. By joint continuity, we have $U_i^m(q,\theta;v^m) > U_i^m(\pi_i^m(q,\theta;v^m),\theta_i;v^m)$ for high enough m, again contradicting the fact that $\pi_i^m(q,\theta;v^m)$ solves $\mathscr{P}_i^m(\mathscr{D},q,\theta;v^m)$. This establishes the claim.

We next claim that for all i and all θ, $\{U_i^m(\cdot,\theta;v^m)\}$ converges uniformly to $U_i^*(\cdot,\theta;v^*)$ on any compact set $Y \subseteq \Re^d$. If not, then there exists $\epsilon > 0$ and a sequence $\{x^m\}$ in Y such that

$$|(1-\delta_i^m)(u_i^m(x^m) + \theta_i \cdot x^m) + \delta_i^m v_i^m(x^m) - (1-\delta_i^*)(u_i^*(x^m) + \theta_i \cdot x^m) - \delta_i^* v_i^*(x^m)| \;\; \ge \;\; \epsilon$$

for all m. By compactness of Y, we may go to a convergent subsequence, still indexed by m, with $x^m \to x \in Y$. But $v^m \to v$ uniformly, and with continuity of our parameterization, we have

$$\begin{aligned}
&\lim_{m\to\infty} (1-\delta_i^m)(u_i^m(x^m) + \theta_i \cdot x^m) + \delta_i^m v_i^m(x^m) \\
&\quad = \;\; (1-\delta_i^*)(u_i^*(x) + \theta_i \cdot x) + \delta_i^* v_i^*(x) \\
&\quad = \;\; \lim_{m\to\infty} (1-\delta_i^*)(u_i^*(x^m) + \theta_i \cdot x^m) + \delta_i^* v_i^*(x^m),
\end{aligned}$$

a contradiction. This establishes the claim.

Finally, let $\hat{v}^m = \psi(v^m, \gamma^m)$ and $\hat{v}^* = \psi(v^*, \gamma^*)$. Let ∂ denote a derivative operator with respect to the coordinates of x of any order $\hat{r} = 0, 1, \ldots, r$. Consider any compact set $Y \subseteq \Re^d$.

We must show that $\partial\hat{v}^m$ converges uniformly to $\partial\hat{v}^*$ on Y. If not, then there exists $\epsilon > 0$, a subsequence $\{\partial\hat{v}^m\}$, still indexed by m, and a corresponding sequence $\{x^m\}$ in Y such that for all m, $||\partial\hat{v}^m(x^m) - \partial\hat{v}^*(x^m)|| \geq \epsilon$. By compactness of Y, we may go to a further subsequence, still indexed by m, such that $x^m \to x$ for some $x \in Y$. Then Aliprantis and Burkinshaw's (1990) Theorem 20.4 implies that for all i and all m,

$$\partial\hat{v}_i^m(x^m) \quad = \quad \int_q \int_\theta \sum_j p_j^m U_i^m(\pi_j^m(q,\theta;v^m),\theta_i;v^m) f^m(\theta)\partial g^m(q|x^m)d\theta dq.$$

Consider the generic case of (q,θ) such that for all j, $\pi_j^m(q,\theta;v^m) \to \pi_j^*(q,\theta;v^*)$. By uniform convergence, from our preceding claim, we have $U_i^m(\pi_j^m(q,\theta;v^m),\theta_i;v^m) \to U_i^*(\pi_i^*(q,\theta;v^*),\theta_i;v^*)$. This gives us pointwise convergence of the integrand of $\partial\hat{v}_i^m(x^m)$ for almost all (q,θ):

$$\sum_j p_j^m U_i^m(\pi_j^m(q,\theta;v^m),\theta_i;v^m) f^m(\theta)\partial g^m(q|x^m) \quad \to \quad \sum_j p_j^* U_i^*(\pi_j^*(q,\theta;v^*),\theta_i;v^*) f^*(\theta)\partial g^*(q|x).$$

Furthermore, since $\partial g^m(q|x^m)$ is zero outside $\tilde{X}$ and since $v^m \in \mathscr{V}$, the terms in the above sequence are bounded in norm by the Lebesgue integrable function $abcI_{\tilde{X}}$. By Lebesgue's dominated convergence theorem, and again using Aliprantis and Burkinshaw's (1990) Theorem 20.4, we therefore have

$$\partial\hat{v}_i^m(x^m) \to \int_q \int_\theta \sum_j p_j^* U_i^*(\pi_j^*(q,\theta;v^*),\theta_i;v^*) f^*(\theta)\partial g^*(q|x)d\theta dq \quad = \quad \partial\hat{v}_i^*(x).$$

By continuity of $\partial\hat{v}_i^*$, we also have $\partial\hat{v}_i^*(x^m) \to \partial\hat{v}_i^*(x)$, but then $|\partial\hat{v}_i(x^m) - \partial\hat{v}_i^*(x^m)| \to 0$. Since i was arbitrary, we have $||\partial\hat{v}^m(x^m) - \partial\hat{v}^*(x)|| \to 0$, a contradiction. We conclude that $\{\partial\hat{v}^m\}$ converges to $\partial\hat{v}^*$ uniformly on Y, and therefore $\hat{v}^m \to \hat{v}^*$, as required. ∎

We can at last turn to the proof of Theorem 1.

Proof of Theorem 1 The statement of Theorem 1 implicitly fixes a model $\gamma \in \Gamma$. By part 1 of Lemma 5, $\mathscr{V}$ is nonempty, convex, and compact. By part 3 of Lemma 5, ψ^γ maps $\mathscr{V}$ to $\mathscr{V}$ and the mapping $\psi^\gamma\colon \mathscr{V} \to \mathscr{V}$ is continuous. Therefore, Glicksberg's theorem yields a fixed point $v^* \in \mathscr{V}$ such that $\psi^\gamma(v^*) = v^*$. We then construct equilibrium strategies as follows: for all i, we specify $\pi_i(q,\theta) = \pi_i^{\mathscr{D}}(q,\theta;v^*)$, and we specify $\alpha_i(y,q,\theta) = 1$ if $y \in A_i(q,\theta;v^*)$ and $\alpha_i(y,q,\theta) = 0$ otherwise. Evidently, the strategy profile $\sigma = (\pi_i,\alpha_i)_{i\in N}$ so defined is a pure stationary legislative equilibrium. Part 1 of Theorem 1 follows from $v^* \in \mathscr{V}$, and parts 2, 3, and 4 follow from part 3 of Lemma 1, part 3 of Lemma 3, and part 1 of Lemma 3, respectively. ∎

The proof of existence in Theorem 1 relied on the fact that every fixed point of ψ^γ corresponds to a stationary legislative equilibrium in model γ. Our final lemma establishes the converse: every pure strategy equilibrium continuation value $v = \psi(v)$. Furthermore, every equilibrium continuation values in γ lie in $\mathscr{V}$.

Lemma 6 *For all $(v,\gamma) \in M(\Re^d,\Re^n) \times \Gamma$, if $v \in E(\gamma)$, then $v \in \mathscr{V}$ and $v = \psi^\gamma(v)$.*

Proof Let $(v,\gamma) \in M(\Re^d,\Re^n) \times \Gamma$ be such that $v \in E(\gamma)$, and let σ be the stationary legislative equilibrium generating v, so that $v = v(\cdot;\sigma)$. As in the proof of part 3 of Lemma 5, define the measurable mapping $w\colon \Re^d \to \Re^n$ by (7) for all i and all q, so that $v(x) = \int_q w(q)g^\gamma(q|x)dq$ for all x. As argued in the proof of part 3 of Lemma 5, part 2 of that lemma then implies that $v \in \mathscr{V}$. Part 3 of Lemma 1 therefore implies that for all i and almost all (q,θ), we have $\pi_i(q,\theta) = \pi_i^\gamma(q,\theta;v)$. This in turn implies that $v = \psi(v,\gamma)$. ∎

We now complete the proofs of the remaining results of the paper.

Proof of Theorem 2 The statement of Theorem 2 implicitly fixes a model $\gamma \in \Gamma$, which we suppress notationally. Consider an arbitrary mixed stationary legislative equilibrium Σ, and let the measurable mapping $v\colon \Re^d \to \Re^n$ be defined by the equilibrium continuation values as $v(x) = (v_1(x;\Sigma), \ldots, v_n(x;\Sigma))$. To facilitate the proof, define

$$W_i(y,q,\theta;\Sigma) \;\; = \;\; \alpha(y,q,\theta;\Sigma)U_i(y,\theta;\Sigma) + (1-\alpha(y,q,\theta;\Sigma))U_i(q,\theta;\Sigma)$$

as the objective function of the proposer given strategy profile Σ.

Now consider any q, and set $\Theta(q) = \Theta_1(q;v) \cup \Theta_4(q;v)$. Consider any $\theta \notin \Theta(q)$. Since $\theta \notin \Theta_1(q;v)$, part 3 of Lemma 1 implies that $\pi_i^{\mathscr{D}}(q,\theta;v)$ is the unique solution to $\mathscr{P}_i(\mathscr{D},q,\theta;v)$. We consider two cases. First, suppose that $\pi_i^{\mathscr{D}}(q,\theta;v) = q$. If we have $\int_{X\setminus\{q\}} \alpha(y,q,\theta;\Sigma)\Pi_i(q,\theta)(dy) > 0$, then there is a set $Y \subseteq X\setminus\{q\}$ such that $\Pi_i(q,\theta)(Y) > 0$ and for all $y \in Y$, $\alpha(y,q,\theta;\Sigma) > 0$. By definition of equilibrium, the latter implies $Y \subseteq A_{\mathscr{D}}(q,\theta;v)$. Then $U_i(q,\theta_i;\Sigma) > U_i(y,\theta_i;\Sigma)$ for all $y \in Y$, which implies

$$W_i(y,q,\theta;\Sigma) \;\; < \;\; U_i(q,\theta_i;\Sigma) \;\; = \;\; W_i(q,q,\theta;\Sigma)$$

for all $y \in Y$, contradicting the fact that Π_i places probability one on maximizers of $W_i(\cdot,q,\theta;\Sigma)$. Therefore, $\int_{X\setminus\{q\}} \alpha(y,q,\theta;\Sigma)\Pi_i(q,\theta)(dy) = 0$. Second, suppose $\pi_i^{\mathscr{D}}(q,\theta;v) \neq q$. We claim that

$$\sup_{y\in X} \; W_i(y,q,\theta;\Sigma) \;\; \geq \;\; U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v).$$

To see this, note that since $\theta \notin \Theta_4(q;v)$, Lemma 4 yields a sequence $\{y^m\}$ in X such that $y^m \to \pi_i^{\mathscr{D}}(q,\theta;v)$ and for all m, there is a decisive coalition C^m satisfying $U_j(y^m,\theta_j,v) > U_j(q,\theta_j;v)$ for all $j \in C^m$. By definition of equilibrium, it then follows that $\alpha_j(y^m,q,\theta_j) = 1$ for all $j \in C^m$, which implies $\alpha(y^m,q,\theta;\Sigma) = 1$. By continuity, we then have $W_i(y^m,q,\theta;\Sigma) = U_i(y^m,\theta_i;v) \to U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v)$, as claimed. Thus, by definition of equilibrium, the mixed proposal strategy Π_i must achieve an expected payoff of at least $U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v)$. Next, we claim that $\Pi_i(\{\pi_i^{\mathscr{D}}(q,\theta;v)\}) = 1$ and $\alpha(\pi_i^{\mathscr{D}}(q,\theta;v),q,\theta;\Sigma) = 1$. Consider any $y \neq \pi_i^{\mathscr{D}}(q,\theta;v)$, and note that $y \notin A_{\mathscr{D}}(q,\theta;v)$ implies $\alpha(y,q,\theta;\Sigma) = 0$, which implies $W_i(y,q,\theta;\Sigma) = U_i(q,\theta_i;v) < U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v)$, a contradiction. And if $y \in A_{\mathscr{D}}(q,\theta;v)\setminus\{q\}$, then $U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v) > \max\{U_i(y,\theta_i;v),U_i(q,\theta_i;v)\}$ implies $W_i(y,q,\theta;\Sigma) < U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v)$, again a contradiction. Therefore, we conclude that Π_i indeed puts probability one on $\pi_i^{\mathscr{D}}(q,\theta;v)$. If $\alpha(\pi_i^{\mathscr{D}}(q,\theta;v),q,\theta;\Sigma) < 1$, then $U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v) > U_i(q,\theta_i;v)$ implies $W_i(\pi_i^{\mathscr{D}}(q,\theta;v),q,\theta;\Sigma) < U_i(\pi_i^{\mathscr{D}}(q,\theta;v),\theta_i;v)$, a contradiction. Thus, we have $\alpha(\pi_i^{\mathscr{D}}(q,\theta;v),q,\theta;\Sigma) = 1$, as claimed.

Finally, we specify pure proposal strategies by $\pi_i(q,\theta) = \pi_i^{\mathscr{D}}(q,\theta;v)$, and we specify pure voting strategies by $\alpha_i(y,q,\theta) = 1$ if $y \in A_i^{\mathscr{D}}(q,\theta;v)$ and $\alpha_i(y,q,\theta) = 0$ otherwise. The pure stationary strategy profile $\sigma = (\pi_i,\alpha_i)_{i\in N}$ generates the same policy outcomes as Σ for almost all (q,θ) and, therefore, the same continuation values. By construction, proposal and voting strategies satisfy the equilibrium conditions of Section 3, and therefore σ is a pure stationary legislative equilibrium. Evidently, Σ is equivalent to σ, and by Lemma 6, the equilibrium continuation value function v lies in $\mathscr{V}$ and is a fixed point of ψ^γ. Then the property of part 1 of Theorem 1 follows immediately, and the properties of parts 2, 3, and 4 follow from part 3 of Lemma 1, part 3 of Lemma 3, and part 1 of Lemma 3, respectively. ∎

Proof of Theorem 3 Consider sequences $\{\gamma^m\}$ in Γ and $\{v^m\}$ in $C^r(\Re^d,\Re^n)$ such that $\gamma^m \to \gamma \in \Gamma$, $v^m \to v \in C^r(\Re^d,\Re^n)$, and for all m, $v^m \in E(\gamma^m)$. By Lemma 6, we have $v^m = \psi(v^m,\gamma^m)$

for all m. Taking limits, we have $v^m \to v$ and, by part 3 of Lemma 5, $\psi(v^m, \gamma^m) \to \psi(v, \gamma)$. Thus, $v = \psi(v, \gamma)$, which implies $v \in E(\gamma)$, establishing closed graph of E. By Lemma 6, the range of ψ lies in $\mathscr{V}$, a compact space, and therefore closed graph of E implies upper hemicontinuity. ∎

Proof of Theorem 4 Let σ^* be a stationary legislative equilibrium. To establish that the operator T satisfies the Feller property, consider any Borel measurable set $Y \subseteq \Re^d$, and let I_Y be the indicator function of Y. Then by arguments in the proof of part 2 of Lemma 5, it follows that the function TI_Y, defined by

$$TI_Y(x) \;=\; P(x, Y) \;=\; \int_q \int_\theta \sum_i p_i I_Y(\pi_i^*(q, \theta)) f(\theta) g(q|x) d\theta dq,$$

is continuous. If $\phi \colon \Re^d \to \Re$ is a simple function, i.e., $\phi(x) = \sum_{l=1}^m a_l I_{Y_l}(x)$ for a measurable partition $\{Y_1, \ldots, Y_m\}$ of $\Re^d$ and coefficients $\{a_1, \ldots, a_m\}$, then $T\phi$ is likewise continuous. Now suppose $\phi \colon \Re^d \to \Re$ is an arbitrary bounded, continuous function. Consider any sequence $\{x^m\}$ in $\Re^d$ with limit x. By Aliprantis and Border's (1999) Theorem 11.6, for all $\epsilon > 0$, there exist simple functions $\phi^1, \phi^2 \colon \Re^d \to \Re$ such that $\phi^1 \leq \phi \leq \phi^2$ and $\int (\phi^2(z) - \phi^1(z)) P(x, dz) < \epsilon$, which implies $T\phi^1 \leq T\phi \leq T\phi^2$ and $T\phi^2(x) - T\phi^1(x) < \epsilon$. By continuity, we have $|T\phi^1(x^m) - T\phi^1(x)| < \epsilon$ and $|T\phi^2(x^m) - T\phi^2(x)| < \epsilon$ for m high enough. Therefore, we have

$$T\phi(x) - T\phi(x^m) \;\leq\; T\phi^2(x) - T\phi^1(x^m) \;=\; T\phi^2(x) - T\phi^1(x) + T\phi^1(x) - T\phi^1(x^m) \;<\; 2\epsilon,$$

with an analogous derivation establishing that $|T\phi(x) - T\phi(x^m)| < 2\epsilon$ for high enough m. Since ϵ is arbitrarily small, it follows that $T\phi$ is continuous. Since $P(x, \tilde{X}) = 1$ for all $x \in \Re^d$, it follows that T is tight. By Futia's (1982) Theorem 2.9, therefore, T admits an invariant distribution, delivering part 1.

For all $x \in X$ and all measurable $Z \subseteq X \times \Theta$, let $Q(x, Z) = \int_Z g(q|x) f(\theta) d(q, \theta)$ denote the probability that next period's (q, θ) lies in Z, conditional on policy choice x this period. To verify Doeblin's condition, define the finite Borel measure η on $\Re^d$ by

$$\eta(Y) \;=\; \int_{\pi_i^{-1}(Y) \cap (\tilde{X} \times \Theta)} f(\theta) d\theta dq.$$

(That is, we integrate the status quo q with respect to Lebesgue measure.) Letting M denote a bound for g, set $\epsilon = \frac{1}{1+M}$. Consider any $x \in \Re^d$ and any measurable $Y \subseteq \Re^d$. Note that $\eta(Y) \leq \epsilon$ implies $M\eta(Y) \leq \frac{M}{1+M}$, and furthermore, we have

$$P(x, Y) \;=\; \sum_{j \in N} p_j Q(x, \pi_j^{-1}(Y)) \;=\; \sum_{j \in N} p_j Q(x, \pi_j^{-1}(Y) \cap (\tilde{X} \times \Theta)) \;\leq\; M\eta(Y),$$

where we use the assumption that the support of $g(\cdot|x)$ lies in $\tilde{X}$. Therefore, $P(x, Y) \leq 1 - \epsilon$, establishing Doeblin. By Futia's (1982) Theorem 4.9, the Markov operator T is quasi-compact, and it follows that the adjoint T^* is also quasi-compact. (See Futia (1982), proof of Theorem 3.3.) For an arbitrary initial distribution μ, Futia's (1982) Theorems 3.2 and 3.4 then yield convergence to an invariant distribution μ^* at the rate claimed in part 2.

Now let i be any legislator with positive recognition probability, $p_i > 0$. By Theorem 2, σ^* satisfies part 3 from Theorem 1, so for all q and almost all θ, π_i^* is continuous at (q, θ). Thus, we can choose $\theta_0 \in \Theta$ such that θ_0 belongs to the support of f and π_i^* is differentiable at (q_0, θ_0). Set $x_0 = \pi_i^*(q_0, \theta_0)$. By continuity of π_i^* at (q_0, θ_0), for every open set $Y \subseteq \Re^d$ containing x_0, there is an open set $Z \subseteq \tilde{X} \times \Theta$ around (q_0, θ_0) such that for all $(q, \theta) \in Z$, we have $\pi_i^*(q, \theta) \in Y$. Given any $x \in \Re^d$, since q_0 lies in the support of $g(\cdot|x)$ and θ_0 lies in the support of f, we have $Q(x, Z) > 0$. We conclude that $P(x, Y) \geq p_i Q(x, Z) > 0$ for all $x \in X$. In fact, if we set $P^1 = P$ and inductively define $P^k(x, Y) = \int P(z, Y) P^{k-1}(x, dz)$, $k = 2, 3, \ldots$, we also have $P^k(x, Y) > 0$ for all $x \in X$ and all $k \geq 1$. Thus, P satisfies the Generalized Uniqueness Criterion 3.5 of Futia (1982). Uniqueness of the invariant distribution now follows by successive application of Futia's Futia (1982) Theorem 3.3 (Feller property and quasi-compactness imply equicontinuity) and 2.12 (equicontinuity and Generalized Uniqueness Criterion imply unique invariant distribution). Finally, Futia's (1982) Theorems 3.6 and 3.7 imply the claimed convergence ratein part 3. ∎

Proof of Theorem 5 Consider a sequence $\{\gamma^m\}$ of quasi-discrete models $\gamma^m = ((p_i, u_i, \delta_i)_{i \in N}, X^m, f, g) \in \Gamma$ such that $\gamma^m \to \gamma = ((p_i, u_i, \delta_i)_{i \in N}, X, f, g) \in \Gamma$. Nonemptiness of $E(\gamma^m)$ follows directly from Theorem 1. Now consider a sequence $\{v^m\}$ of equilibrium continuation values such that $v^m \in E(\gamma^m)$ for all m. By Lemma 6, we have $v^m = \psi(v^m, \gamma^m) \in \mathscr{V}$ for all m. By compactness of $\mathscr{V}$, from part 1 of Lemma 5, $\{v^m\}$ admits a convergent subsequence with limit, say, v. Then Theorem 3 implies $v \in E(\gamma)$. ∎

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